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TECHNICAL UNIVERSITY OF DENMARK

M. W. Bræstrup

YIELD – LINE THEORY AND LIMIT ANALYSIS OF PLATES AND SLABS

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Yield-line theory and limit analysis of plates and slabs

Civilingeniør M. W. Bræstrup

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SYNOPSIS

The usual geometrical, statical and physical conditions for the anisotropic, perfectly plastic plate are given. It is demonstrated that the normal moment criterion usually employed in modern yield-line theory is identical with the 'stepped' criterion of the classical theory, and that they both correspond to a yield surface, called the upper yield surface, which satisfies the requirements of limit analysis. When the actual yield surface of the plate is different from the upper yield surface, then the yield load predicted by limit analysis can generally not be determined by yield-line theory. It is not even possible to approach the solution by successive refinement of the yield-line pattern. The common yield surfaces of metal plates are compared with the upper yield surface, and the yield surface of the arbitrarily reinforced concrete slab is derived. Finally, the relationship between yield-line theory and limit analysis is discussed, and it is concluded that the two theories are consistent in their foundation.

Introduction

Ever since the mathematical theory of plasticity received its strict formulation, it has been generally agreed upon that the yield-line theory of Johansen[1] may be placed in the framework of limit analysis. Following Prager[2], yield-line theory is considered as a simple and rapidly converging method to determine upper bounds of the yield load of plates and slabs. During recent years, however, it has been stated by the British authors Wood[3,4] and Jones and Wood[5] that yield-line theory and limit analysis are inconsistent. Arguments for this point of view are provided by the somewhat different physical conditions that are used in the two theories. Indeed, limit analysis rests on a yield condition expressed in the two bending moments and the twisting moment (or alternatively in the two principal moments), whilst yield-line theory employs a yield criterion involving only the bending moment in the yield lines.

It is the aim of this investigation to demonstrate that these controversies are but apparent, and that the yield-line theory in its usual form rests firmly on the ground of limit analysis. At the same time the paper examines in which cases application of the yield-line theory may lead to a yield load which is correct in the sense of limit analysis.

Notation

The symbols are defined when they first occur. The following list gives those which are used repeatedly.

A, B, C	= constants of the upper yield surface (positive yielding)
a, b, c	= constants of the upper yield surface (negative yielding)
$M(\varphi)$	= value of M_n in a positive yield section
$m(\varphi)$	= value of $-M_n$ in a negative yield section
M_n	= bending moment per unit length of the section with normal n
M_{ns}	= twisting moment per unit length of the section with normal n and tangent s
M_o	= average yield moment (positive yielding)
m_o	= average yield moment (negative yielding)
M_i	= positive yield moment of the ith band of reinforcement
m_i	= negative yield moment of the ith band of reinforcement
n, s	= axes of the local co-ordinate system, n being the normal and s the tangent of the considered section
p	= distributed load per unit area (load parameter)
Q_n	= shear force per unit length of the section with normal n

R	= radius of anisotropy (positive yielding)
r	= radius of anisotropy (negative yielding)
$T(\varphi)$	= value of M_{ns} in a positive yield section
$t(\varphi)$	= value of $-M_{ns}$ in a negative yield section
$\dot{w}$	= rate of deflexion
x, y, z	= axes of the fixed co-ordinate system
α_i	= angle from the x axis to the ith band of reinforcement
$\dot{\kappa}_n$	= rate of curvature in direction of the axis n
$\dot{\kappa}_{ns}$	= rate of twist of the axes n and s
$\dot{\theta}_n$	= rate of slope discontinuity in the section with normal n
φ	= angle from the x axis to the n axis and from the y axis to the s axis

Basic assumptions

The plate is considered as a two-dimensional body occupying the connected region A in the plane $z = 0$ of the fixed orthogonal Cartesian co-ordinate system with axes x, y and z. The exterior and interior boundary curves are termed L, and curves along which the deformations are discontinuous are labelled S. Points on an arbitrary curve are assigned a local co-ordinate system with axes n and s, normal and tangential to the curve, respectively (see Figure 1). The orientation of the local system is determined by the angle φ, which is the angle from the x axis to the n axis and from the y axis to the s axis. The arc differential ds of the curve is positive in the direction of the s axis.

The influence of the deformations on the geometrical, statical and physical conditions is neglected. Further, the geometrical and statical assumptions are those generally adopted in plate theory. Thus the virtual work equation may be expressed as follows:

$$\int_A p\dot{w}\,dA + \Sigma\int_L \left[\dot{w}\left(Q_n - \frac{\partial M_{ns}}{\partial s}\right) - \frac{\partial \dot{w}}{\partial n}M_n\right] ds$$
$$= \int_A (\dot{\kappa}_x M_x + \dot{\kappa}_y M_y + 2\dot{\kappa}_{xy} M_{xy})\,dA + \Sigma\int_S (\dot{\theta}_n \cos^2\varphi M_x + \dot{\theta}_n \sin^2\varphi M_y - 2\dot{\theta}_n \cos\varphi \sin\varphi M_{xy})\,ds \quad \ldots\ldots\ldots(1)$$

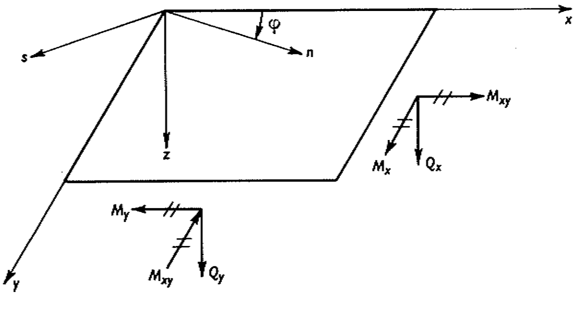

Figure 1: Plate element with stress resultants.

Here $p = p(x,y)$ and $\dot{w} = \dot{w}(x,y)$ are the distributed load per unit area and the deflexion rate, both positive in the direction of the z axis. The magnitude of the load is determined by a factor called the load parameter, the distribution being fixed.

The positive directions of the stress resultants referred to the fixed axes are shown in Figure 1. These resultants and the load fulfil the equilibrium equations:

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + p = 0$$

$$\frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} - Q_x = 0$$

$$\frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} - Q_y = 0$$

The stress resultants of an arbitrary section with the normal n are determined by the transformation formulas:

$$M_n = M_x \cos^2\varphi + M_y \sin^2\varphi - 2M_{xy}\cos\varphi\sin\varphi \quad \ldots\ldots(2)$$

$$M_{ns} = (M_x - M_y)\cos\varphi\sin\varphi + M_{xy}(\cos^2\varphi - \sin^2\varphi) \quad \ldots\ldots(3)$$

$$Q_n = Q_x \cos\varphi + Q_y \sin\varphi$$

The curvature rates referred to the fixed axes are defined as:

$$\dot{\kappa}_x = -\frac{\partial^2 \dot{w}}{\partial x^2},\quad \dot{\kappa}_y = -\frac{\partial^2 \dot{w}}{\partial y^2},\quad \dot{\kappa}_{xy} = \frac{\partial^2 \dot{w}}{\partial x \partial y}$$

For curves along which the deflexion rate is not twice differentiable, the quantity $\dot{\theta}_n$ is introduced by:

$$\theta_n = \left(\frac{\partial w}{\partial n}\right)_- - \left(\frac{\partial w}{\partial n}\right)_+$$

Thus $\dot{\theta}_n$ is the rate of the jump in the slope when the discontinuity is crossed with decreasing n. The curve of discontinuity is called a yield line, and it is termed positive or negative according to the sign of $\dot{\theta}_n$.

The generalized stresses and strain rates are statical and geometrical quantities such that the sum of their products constitute the internal work by a virtual deformation. The internal work for the plate, i.e. the expression on the right-hand side of equation 1, consists of two parts, each of which gives rise to the definition of a set of generalized variables. In both cases, the moments M_x, M_y and M_{xy} are chosen as the generalized stresses. The corresponding generalized strain rates are then $\dot{\kappa}_x$, $\dot{\kappa}_y$ and $2\dot{\kappa}_{xy}$ outside the yield lines and $\dot{\theta}_n \cos^2\varphi$, $\dot{\theta}_n \sin^2\varphi$ and $-2\dot{\theta}_n \cos\varphi\sin\varphi$ in the yield lines. The two types of generalized strain rates are respectively called distributed and concentrated.

Physical conditions

It is assumed that the plate may be considered as a rigid, perfectly plastic body. The yield load is then the least value of the load parameter at which deformations may occur.

A yield function $f(M_x,M_y,M_{xy})$ is a differentiable function of the generalized stresses with the property that deformations may be present for $f = 0$ whereas stress states for which $f > 0$ are impossible. The yield condition of the plate is a set of inequalities of the form:

$$f_i(M_x,M_y,M_{xy}) \leqq 0$$

where f_i is a finite or infinite number of yield functions. The yield functions must have such a character that the stress states satisfying the yield condition occupy a region in (M_x,M_y,M_{xy}) space, bounded by the surface:

$$F(M_x,M_y,M_{xy}) = 0$$

where F is a continuous, piecewise differentiable function of the generalized stresses. The surface $F = 0$ is called the yield surface. It is required that through every point of the surface there exists at least one plane, such that the surface lies entirely on one side of the plane (the convexity condition). The plane is called a supporting plane of the surface.

According to the yield condition, deformations can only be present when the stress point is on the yield surface. The relationship between the generalized strain rates and the generalized stresses is then given by the flow law. This law requires that the strain vector $(\dot{\kappa}_x, \dot{\kappa}_y, 2\dot{\kappa}_{xy})$ or $(\dot{\theta}_n \cos^2 \varphi, \dot{\theta}_n \sin^2 \varphi, -2\dot{\theta}_n \cos \varphi \sin \varphi)$ is normal to a supporting plane through the stress point and directed away from the region of permitted stress states (the normality condition).

It is assumed that the yield surface for a point of the plate is the same, whether the strain rates are concentrated or distributed. In the former case, however, the physical conditions may be given in a considerably simpler form if the quantity $\dot{\theta}_n$ is chosen as generalized strain rate. It then follows from equations 1 and 2 that the corresponding generalized stress is M_n. Hence the yield condition reduces to:

$$-m(\varphi) \leqq M_n \leqq M(\varphi) \quad \ldots\ldots\ldots\ldots(4)$$

The yield surface and the flow law in one-dimensional stress space may be visualized as shown in Figure 2.

Before use can be made of the yield condition, the yield moments M and m must be known as functions

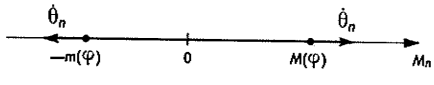

Figure 2: Yield surface and flow law for yield lines.

of the angle φ. These functions may be plotted as so-called polar diagrams, i.e. curves $r = M(\varphi)$ and $r = m(\varphi)$ where (r, φ) are polar co-ordinates.

Evidently the polar diagrams must bear some relationship to the yield surface of the plate. To investigate this, a point on a positive yield line is considered. The stress state is assumed given by the point $(M_{xo}, M_{yo}, M_{xyo})$ which, according to the yield condition, lies on the yield surface. The strain-rate vector is $(\dot{\theta}_n \cos^2 \varphi, \dot{\theta}_n \sin^2 \varphi, -2\dot{\theta}_n \cos \varphi \sin \varphi)$ and a plane orthogonal to this vector has the equation:

$$\dot{\theta}_n \cos^2 \varphi\,(M_x - M_{xo}) + \dot{\theta}_n \sin^2 \varphi\,(M_y - M_{yo}) \\ -2\dot{\theta}_n \cos \varphi \sin \varphi\,(M_{xy} - M_{xyo}) = 0$$

The flow law implies that this is a supporting plane to the yield surface. However, the stress state also satisfies the normal moment criterion:

$$M_{no} = M_{xo} \cos^2 \varphi + M_{yo} \sin^2 \varphi \\ -2M_{xyo} \cos \varphi \sin \varphi = M(\varphi)$$

Introducing this and dividing by $\dot{\theta}_n$ we get:

$$M_x \cos^2 \varphi + M_y \sin^2 \varphi \\ -2M_{xy} \cos \varphi \sin \varphi - M(\varphi) = 0 \quad \ldots\ldots(5)$$

For negative yield lines, we obtain analogously:

$$M_x \cos^2 \varphi + M_y \sin^2 \varphi \\ -2M_{xy} \cos \varphi \sin \varphi + m(\varphi) = 0 \quad \ldots\ldots(6)$$

It follows that the relationship between the yield surface and the polar diagrams consists in the requirement that each of the planes given by equations 5 and 6 be a supporting plane to the surface for any value of φ. This has been shown by Massonnet and Save[6] and also used by Save[7].

Following Kemp[8] the condition 4 with associated polar diagrams is called the normal moment criterion, and it is the yield criterion which is used in yield-line theory. It appears that the form of the criterion is fully consistent with the assumptions of limit analysis but, because of the dependence developed above, it may be expected that the existence of a continuous and convex yield surface imposes some restrictions on the polar diagrams. This point is examined in the following section.

Polar diagrams

We have seen that the polar diagrams define two families of planes in M_x, M_y, M_{xy} space. The plate will be a perfectly plastic body if there exists a convex surface that touches all of these plates.

A necessary, but generally not sufficient, condition is that the planes envelop a surface. Indeed, if this is not the case, then the line of intersection between two neighbouring planes corresponding to parameters φ and $\varphi + \Delta\varphi$ will not have a limit position as $\Delta\varphi \rightarrow 0$. Hence there will be no continuous surface that touches both planes.

A sufficient, but generally not necessary, condition is that the envelope is convex. In this case, the envelope will be a possible yield surface, satisfying the conditions of convexity and normality.

From the differential geometry it is known that the envelope of a one-parameter family of planes is either a cylindrical, a conical, or a tangential surface. Of these three possibilities, the first may be excluded by mere inspection of equations 5 and 6.

If the envelope is conical, then the planes have a common point of intersection which is the vertex of the cone. This point is labelled (A, B, C) for the positive yield lines and $(-a, -b, -c)$ for the negative yield lines. From equations 5 and 6, we then see that the polar diagrams are:

$$M(\varphi) = A\cos^2\varphi + B\sin^2\varphi - 2C\cos\varphi\sin\varphi \quad \ldots(7)$$

$$m(\varphi) = a\cos^2\varphi + b\sin^2\varphi - 2c\cos\varphi\sin\varphi \quad \ldots(8)$$

As we shall see in the next section, the conical envelope is, in fact, convex. Therefore we have demonstrated the following:

Theorem I: Polar diagrams of the form of equations 7 and 8 are consistent with the physical conditions of a perfectly plastic material.

The third possibility, the tangential surface, is not convex. This means that, if the polar diagrams differ from equations 7 and 8, it requires a closer examination to determine whether they may be compatible with limit analysis.

In the following, equations 7 and 8 are taken as the polar diagrams of the plate. Comparing with equation 2, we see that the yield moments transform as the normal moments, i.e. according to Mohr's circle. Therefore the normal moment criterion may be represented graphically as shown in Figure 3, where the values of $M(\varphi)$ and $m(\varphi)$ are found as the abscissae to the corresponding points on the two circles. The radius from which the angle 2φ has to be measured depends upon the choice of fixed co-ordinate system. The two circles are the Mohr circles for points with positive or negative yielding in all sections. Comparing with equation 3, the value of the twisting moment is found to be $M_{ns} = T(\varphi)$ or $M_{ns} = -t(\varphi)$ for positive and negative yielding, respectively, where:

$$T(\varphi) = (A - B)\cos\varphi\sin\varphi + C(\cos^2\varphi - \sin^2\varphi) \quad \ldots(9)$$

$$t(\varphi) = (a - b)\cos\varphi\sin\varphi + c(\cos^2\varphi - \sin^2\varphi) \quad \ldots(10)$$

This is in agreement with the formulas given by Johansen[1]. Hence, with polar diagrams given by equations 7 and 8, the normal moment criterion is identical with Johansen's 'stepped' criterion. These polar diagrams are the ones that are usually employed in yield-line analysis.

The representation provided by Figure 3 combines the 'yield surface' of Figure 2 with the information contained in the polar diagrams. It is specially suited for the derivation of the statical conditions for the yield lines, i.e. the rules governing the intersection of yield lines with other yield lines and with the boundary (details may be found in reference 9).

The quantities A, B, C and a, b, c are material constants, but they depend upon the orientation of the co-ordinate system. As invariant strength parameters, we propose the quantities shown in Figure 3:

$$M_o = \tfrac{1}{2}(A + B),\quad R^2 = \tfrac{1}{4}(A - B)^2 + C^2$$

$$m_o = \tfrac{1}{2}(a + b),\quad r^2 = \tfrac{1}{4}(a - b)^2 + c^2$$

They may conveniently be termed the average yield moments and the radii of anisotropy. The strength parameters contain the physical information necessary in the yield-line theory. They do not define the material in the sense of limit analysis because the yield surface is only partially determined.

The upper yield surface

Consider a plate for which the strength parameters defined in the preceding section are known. The surface, which is enveloped by the families of planes given by equations 5 and 6, now has to be determined. The derivation is similar to that of Save[7], the only difference being that it is not restricted to orthotropic plates. First the positive yield lines are considered.

The equation of the envelope is obtained by elimination of the parameter φ between the equation of the planes and the equation derived by differentiation with respect to φ. Equations 5 and 7 give:

$$(M_x - A)\cos^2\varphi + (M_y - B)\sin^2\varphi = 2(M_{xy} - C)\cos\varphi\sin\varphi \quad \ldots(11)$$

or

$$(M_x - A)\cot\varphi + (M_y - B)\tan\varphi = 2(M_{xy} - C)$$

Differentiating and multiplying by $\cos^2\varphi\sin^2\varphi$, we get:

$$-(M_x - A)\cos^2\varphi + (M_y - B)\sin^2\varphi = 0 \quad \ldots(12)$$

Finally, equations 11 and 12 are squared and subtracted to give:

$$(M_x - A)(M_y - B) = (M_{xy} - C)^2$$

This is the equation of a cone with vertex in $(M_x, M_y, M_{xy}) = (A, B, C)$, axis parallel to the plane $M_{xy} = 0$

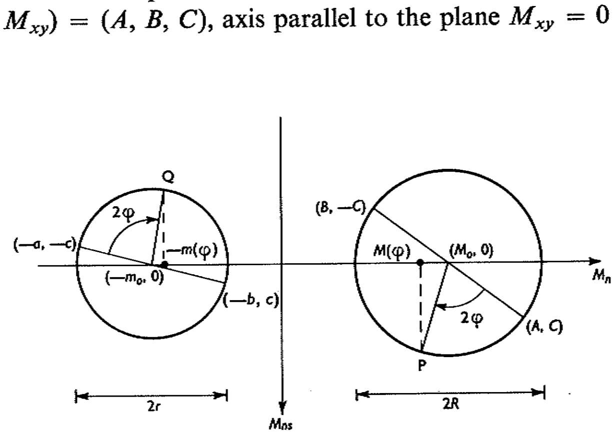

Figure 3: The normal moment criterion.

and elliptic normal sections. The projections on the plane $M_{xy} = 0$ are the lines $M_x = A$ and $M_y = B$.

In the same way we obtain from equations 6 and 8:

$$(M_x + a)(M_y + b) = (M_{xy} + c)^2$$

With appropriate restrictions on the constants, these two cones will form a closed, convex surface, enclosing the origin, and enveloping both the families of planes. Hence a possible yield surface is any surface which is inscribed in the bi-conical envelope in such a way that all the generatrices are touched. As the envelope circumscribes all the other surfaces, it will be termed the upper yield surface of the plate corresponding to the given strength parameters.

The upper yield surface is shown in Figure 4. Points on the cones PSTRU and QSTRU correspond to positive and negative yield lines or to regions with zero gaussian curvature rate. The vertices P and Q correspond to the intersection of yield lines of the same sign or to regions with a positive gaussian curvature rate. Finally, the inclined ellipse STRU corresponds to the intersection of yield lines of opposite sign or to regions with a negative gaussian curvature rate.

The validity of yield-line solutions

The idea of yield-line theory is to choose a displacement field where the strain rates are concentrated in yield lines while the rest of the plate remains rigid. Thus upper bounds for the yield load are determined. The question is, under what circumstances can such a procedure lead to the correct yield load?

The moment state of a point on a yield line is described by a point on the upper yield surface or, more correctly, a point on the generatrix along which the corresponding tangent plane touches the conical surface. However, the point must also lie on the actual yield surface, i.e. we have:

Theorem II: A solution involving yield lines can only be correct if the points on the upper yield surface, corresponding to the yield sections, lie on the actual yield surface.

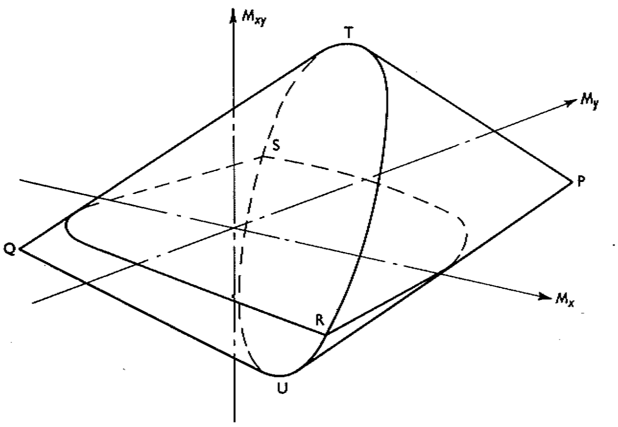

Figure 4: The upper yield surface.

This rather obvious statement is not handy for practical use because the moment state is not determined by the direction of the yield line. The normal moment may be found from equation 7 or 8 and the twisting moment is given by equation 9 or 10, but the tangential bending moment is arbitrary. This corresponds to the degree of freedom for the stress point which may lie anywhere on the generatrix. However, in the case of a homogeneous plate where all points have the same yield surface, we may draw some useful conclusions. It is noted that, if solutions involving intersecting yield lines of the same sign are to be possible, the actual yield surface must contain the regions at the vertices P and Q (Figure 4). Analogously the region about the ellipse STRU allows the intersection of yield lines of opposite sign. Finally the convexity rule implies that, if both types of intersection are to be possible, the actual yield surface must be identical with the upper yield surface.

It is tempting to presume that, whatever the form of the actual yield surface, a successive refinement of the yield-line pattern will lead to an arbitrarily close measure of the yield load. Indeed, any surface of deflexion may be approximated as closely as desired by a net of yield lines. Nevertheless, the assumption is not correct and the following limitation is valid:

Theorem III: The smallest upper bound which may be found by yield-line theory is the exact solution corresponding to the upper yield surface.

The truth of theorem III follows from the fact that the yield-line solution depends only upon the strength parameters, and not upon the form of the actual yield surface. The solution will, therefore, also be an upper bound if the actual yield surface is replaced by the upper yield surface and accordingly cannot be less than the correct solution corresponding to that case. An example of the application of theorem III will be given in the next section.

It is a consequence of theorems II and III that it is not possible to determine the yield load of the plate by yield-line theory if the mentioned correct solution involves parts of the upper yield surface which are not lying on the actual yield surface. This limitation does not seem to be generally known.

In practice, yield-line theory is useful also for plates with a yield surface different from the upper yield surface. It has even been used in cases where the yield surface has as yet not been known, e.g. the skewly reinforced concrete slabs which will be considered later. An estimate of the inaccuracy that must be envisaged can be obtained by comparing the actual yield surface with the upper yield surface.

Isotropic metal plates

The yield conditions usually applied for metal plates are:

$$\max(|M_1|, |M_2|, |M_1 - M_2|) - M_o \leqq 0 \quad \text{(Tresca)}$$

and $M_x^2 - M_xM_y + M_y^2 + 3M_{xy}^2 - M_o^2 \leqq 0$ (von Mises)

Here M_1 and M_2 are the principal moments and $M_o = \frac{1}{4}\sigma_o h^2$, where h is the plate thickness and σ_o is the yield stress of the material (equal for tension and compression). The polar diagrams are:

$$M(\varphi) = m(\varphi) = M_o \quad \text{(Tresca)}$$

and

$$M(\varphi) = m(\varphi) = 2/\sqrt{3}M_o = M_o^* \quad \text{(von Mises)}$$

The intersection of the two yield surfaces with the plane $M_{xy} = 0$ is shown on Figure 5 and the corresponding upper yield surfaces are indicated by dotted lines. It is noted that the Tresca surface partially coincides with the upper yield surface, whilst this is not the case with the von Mises surface which only touches the upper yield surface along two ellipses.

As an illustration of the resulting difference between solutions corresponding to the two yield conditions, consider a homogeneous plate of constant thickness, one-way spanning between two simple supports (Figure 6). The load is uniformly distributed of intensity p per unit area.

Using the Tresca condition and assuming a yield line in $x = 0$, we obtain:

$$p = 8M_o/l^2 \quad \text{(Tresca)}$$

This is the correct solution, because if the statically admissible moment field:

$$M_x = M_o(1 - 4x^2/l^2),\ M_y = M_{xy} = 0$$

is inserted in the equilibrium equations, the same value of p is found.

If the von Mises condition is valid, bounds for the yield load may be obtained by consideration of the inscribed and the circumscribed Tresca-surface. We get:

$$8M_o/l^2 < p < 8M_o^*/l^2 \quad \text{(von Mises)}$$

The upper bound corresponds to a yield line in $x = 0$. The moment state in the yield line is given by the point A on Figure 5, which is incompatible with the boundary condition $M_y = 0$ for $y = \pm\frac{1}{2}\rho l$. Hence the solution is only correct at the limit $\rho \rightarrow \infty$. The lower bound corresponds to the moment field given above, the moment state in the line $x = 0$ being described by the point B of Figure 5. The flow law then requires $\varkappa_y \neq 0$, inconsistent with the fact that only the points $x = 0$ are on the yield surface. Therefore also this solution is incorrect except at the limit $\rho \rightarrow 0$. When ρ traverses the interval $0 < \rho < \infty$, the yield load traverses the interval given above.

It follows from theorem III that any attempt to improve the upper bound for the von Mises plate, using more complicated yield-line patterns, is bound to fail. Indeed, if a value $p = kM_o^*/l^2$ with $k < 8$ was obtained, then the same pattern would give $p = kM_o/l^2$ for the Tresca plate, i.e. a value less than the correct one. Obviously yield-line theory fails to provide the correct solution in this case, but a fairly good approximation is found. This is due to the fact that the von Mises condition does not differ too much from the upper yield surface in the region of interest. For plates with pronounced twisting, the results would be less valuable also for Tresca plates.

This very trivial example shows how yield conditions that permit yield lines lead to less complicated solutions. Further, it is worthy of note that the yield load for the von Mises plate considered depends upon the width-to-span ratio ρ. This may serve as a simple experimental guide as to which condition best describes reality.

Anisotropic reinforced concrete slabs

Consider a point of the slab with reinforcement in directions α_i with intensities corresponding to the positive yield moments M_i and the negative yield moments m_i (Figure 7). For our purpose, the calculation of M_i and m_i need not be specified further. It is then usual to assume (Johansen[1], Jones and Wood[5]) that the polar diagrams are:

$$M(\varphi) = \sum_i M_i \cos^2(\varphi - \alpha_i)$$

and

$$m(\varphi) = \sum_i m_i \cos^2(\varphi - \alpha_i)$$

It is easily seen that these functions are of the form referred to in theorem I, with:

$$A = \sum_i M_i \cos^2\alpha_i,\quad B = \sum_i M_i \sin^2\alpha_i,$$

$$C = -\sum_i M_i \cos\alpha_i \sin\alpha_i$$

$$a = \sum_i m_i \cos^2\alpha_i,\quad b = \sum_i m_i \sin^2\alpha_i,$$

$$c = -\sum_i m_i \cos\alpha_i \sin\alpha_i$$

The strength parameters are:

$$M_o = \tfrac{1}{2}\sum_i M_i,\quad R^2 = \tfrac{1}{4}\sum_{i,j} M_iM_j \cos 2(\alpha_i - \alpha_j)$$

$$m_o = \tfrac{1}{2}\sum_i m_i,\quad r^2 = \tfrac{1}{4}\sum_{i,j} m_im_j \cos 2(\alpha_i - \alpha_j)$$

The yield condition of the slab may be taken as the normal moment criterion if it is assumed to be valid not only in the yield lines, but everywhere in the slab. In words the yield condition may be expressed as follows: if the bending moment M_n in a section with the normal n attains the corresponding yield value $M(\varphi)$ or $-m(\varphi)$, then strain rates $\dot{\varkappa}_n$ or $\dot{\theta}_n$ may be present in the section. If also neighbouring sections are at yield the strain rates may be distributed, otherwise they are concentrated. More formally the yield condition is written:

$$f_\varphi(M_x, M_y, M_{xy}) \leqq 0,\ g_\varphi(M_x, M_y, M_{xy}) \leqq 0,$$

where the infinity of yield functions f_φ and g_φ corresponds to the supporting planes, i.e.:

$$f_\varphi = (M_x - A)\cos^2\varphi + (M_y - B)\sin^2\varphi - 2(M_{xy} - C)\cos\varphi\sin\varphi$$

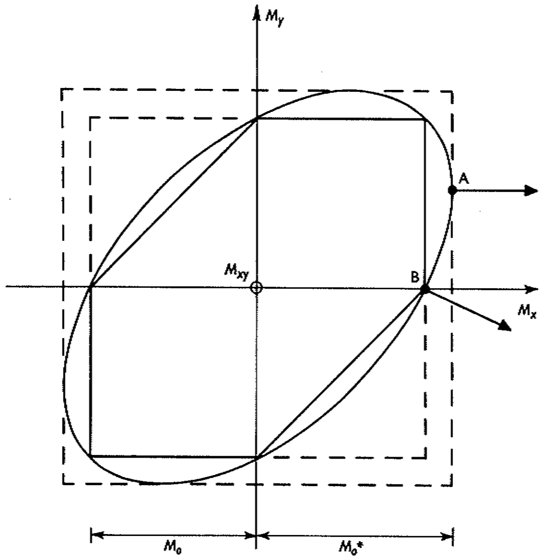

Figure 5: Yield surfaces of Tresca and von Mises.

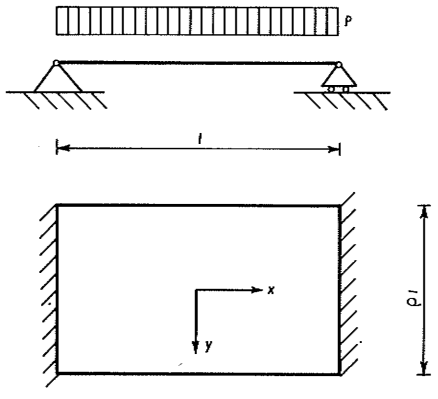

Figure 6: Example: one-way-spanning plate with simple supports.

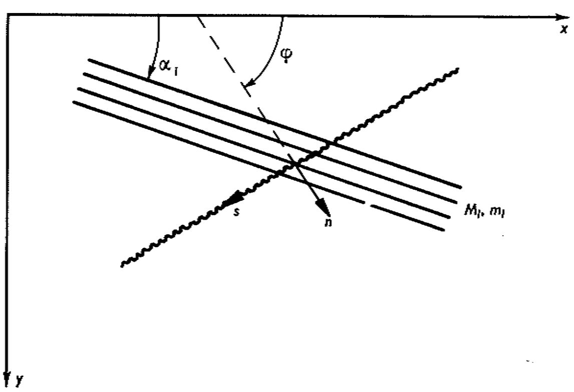

Figure 7: Band of reinforcement crossing a yield line.

$$g\varphi = -(M_x + a)\cos^2\varphi - (M_y + b)\sin^2\varphi + 2(M_{xy} + c)\cos\varphi\sin\varphi$$

The yield surface becomes identical with the upper yield surface of the plate (Figure 4). Therefore the yield condition may also be expressed:

$$\left.\begin{aligned}(M_{xy} - C)^2 - (M_x - A)(M_y - B) &\leqq 0\\ (M_{xy} + c)^2 - (M_x + a)(M_y + b) &\leqq 0\end{aligned}\right\} \quad ..(13)$$

with the restrictions:

$$-a \leqq M_x \leqq A, \quad -b \leqq M_y \leqq B$$

According to the 'stepped' criterion of Johansen[1], the twisting moment in a positive yield line is given as:

$$T(\varphi) = \sum_i M_i \cos(\varphi - \alpha_i)\sin(\varphi - \alpha_i)$$
$$= (A - B)\cos\varphi\sin\varphi + C(\cos^2\varphi - \sin^2\varphi)$$

in agreement with equation 9. The same result and the corresponding expression for $t(\varphi)$ may also be found from consideration of the upper yield surface.

Whilst the yield condition in terms of the normal moments is familiar, the form 13 and hence the shape of the yield surface have hitherto only been known in the isotropic and orthotropic cases. The slab is said to be orthotropically reinforced when the reinforcement has two orthogonal axes of symmetry. If these directions are taken as the axes of the co-ordinate system, both the constants C and c take the value zero. The corresponding yield surface with both vertices in the plane $M_{xy} = 0$ is derived by Nielsen[10] from consideration of a plate element. Save[7] obtains the surface from the normal moment criterion and the polar diagrams. Kemp[8] also uses the normal moment criterion, but the resulting condition is expressed in a co-ordinate system with axes in the principal moment directions; hence C and c are different from zero. The yield curve in the principal moment plane $M_{xy} = 0$ depends upon the angle between the principal moment axes and the axes of symmetry for the reinforcement. In general it is formed by two hyperbolas (cf. Figure 4).

In the case when the plate is isotropic we get $A = B = M_o$, $a = b = m_o$, and $C = c = 0$ for any orientation of the co-ordinate system. The curve of intersection with the principal moment plane is the well-known square yield locus.

Discussion and conclusion

We are now in a position to examine the contradictions which, it is claimed, exist between yield-line theory and limit analysis. The arguments have been summed up by Jones and Wood[5] for the isotropic slab as follows.

(1) The 'stepped' criterion insists on a specific value of the tangential bending moment M_s; this is not the case with the 'square' criterion.

(2) The flow law and the 'square' criterion require that positive and negative yield lines intersect each

other at right-angles. This limitation is not valid in yield-line theory.

(3) The normal moment criterion specifies only the normal bending moment, the 'square' criterion requires that the twisting moment is zero.

Objection (1) is simply not correct and must arise from a misunderstanding. On the contrary, it is the freedom of the tangential moment which is the clue to the simplicity of yield-line analysis. Item (2) refers to one of the statical conditions for the yield lines that may be derived on the very basis of yield-line theory (cf. Braestrup[9]). In the isotropic case, they are familiar and are discussed by Johansen[1]. The fact that yield-line analysis often involves statically inadmissible yield-line patterns only reflects that it is an upper-bound technique. Finally, referring to (3), we have shown that, with the usual polar diagrams, the normal moment criterion and the upper yield surface are different expressions for the same physical conditions. The normal moment criterion is the most practical for upper-bound determination by yield-line analysis because it only involves one stress resultant. On the other hand, to determine whether or not a given moment field is statically admissible, it is more convenient to use the equation of the yield surface than to test all sections against the normal moment criterion.

Jones and Wood[5] write as follows concerning the normal moment criterion: "Such a criterion is useless within the strict framework of limit analysis, which must develop its own idealized criteria of yield. Until yield-line theory and limit analysis employ the same criterion of yield, they must go their own separate ways." This conclusion appears to be unfounded and, in our view, it would be unfortunate if it were not corrected.

This does not mean that yield-line analysis and upper-bound determination for plates and slabs are identical concepts. We have seen that, when the yield surface is different from the upper yield surface, yield-line theory is not sufficient, and it may be profitable to search upper bounds by other methods. On the other hand, yield-line theory gives an approximate solution to the entire plate problem and not only a value of the yield load. At any rate, it would be deplorable if only the differences were noted, and the two theories were allowed to drift apart without their mutual connexion being explored.

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