AFDELINGEN FOR

BÆRENDE KONSTRUKTIONER

DANMARKS TEKNISKE HØJSKOLE

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LINEARLY VISCOELASTIC FINITE ELEMENTS

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1. Introduction

The problem of the determination of the stresses and displacements in a structure which takes viscoelastic effects into account will only have analytical solutions in some simple cases, Elias [15]. More complicated problems must generally be solved by numerical methods.

A summerized treatment of the theory of viscoelasticity is given by Christensen [1]. The foundation of the theory applied to this paper is given by Gurtin & Sternberg [3]. It can be shown that the constititive equations contain a discription of the well-known Maxwell-, Kelvin- and Burger-materials, Rabotnov [2].

A general and useful method is the finite element method, White [4]. Two different types of elements will be derived from functionals given by Gurtin [6] by variational principles.

Another numerical method is based on difference equations. This method is usually preferred when the problem depends on one dimension. Solutions to such problems involving concrete structures are given by Wissmann [12] and Aagaard Sørensen & Højlund Rasmussen [13].

2. Symbols and assumptions

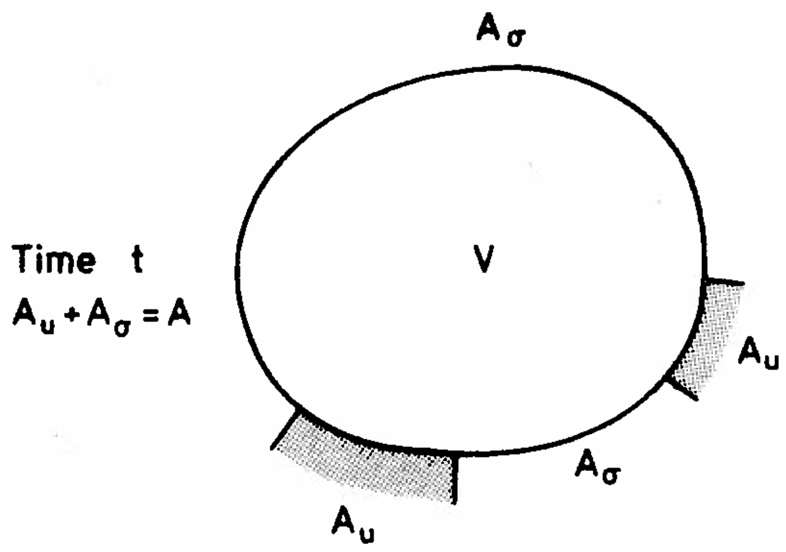

Fig. 1

The following symbols are used:

$G_{ijkl}(t)$: The components of the relaxation tensor, expressing the elastic memory of the material. It is the visco-elastic analogy to the elasticity tensor.

$J_{ijkl}(t)$: The components of the creep tensor. The creep tensor is the viscoelastic analogy to the compliance tensor.

$\varepsilon_{ij}(t)$: The components of the strain tensor.

$\sigma_{ij}(t)$: The components of the stress tensor.

$u_i(t)$: The components of the displacement vector.

$T_i(t)$: The components of the surface traction vector.

$F_i(t)$: The components of the body force vector.

$\bar{u}_i(t)$, $\bar{T}_i(t)$ and $\bar{F}_i(t)$ are the prescribed values of the displacements, the surface tractions and the body forces.

All functions are assumed to vanish for $t < 0$.

The constitutive conditions are expressed by

$$\sigma_{ij} = G_{ijkl} * d\epsilon_{kl} \tag{1}$$

and $$\epsilon_{ij} = J_{ijkl} * d\sigma_{kl} \tag{2}$$

where "*" denotes Stieltjes' convolution product, see appendix. A combination of (1) and (2) gives the relation between the creep and the relaxation tensor.

$$J_{ijkl} * dG_{klmn} = h(t)\delta_{im}\,\delta_{jn} \tag{3}$$

where h(t) is the unit step function, see appendix.

The following is assumed concerning the relaxation tensor and, by means of (3), also concerning the creep tensor:

$$G_{ijkl} = G_{jikl} = G_{ijkl} \tag{4}$$

$$\lim_{t\to\infty} G_{ijkl} \text{ exists} \tag{5}$$

$$G_{ijkl} \text{ is continuous for } t \in [0,\infty[\tag{6}$$

$$G_{ijkl}(t) = G_{ijkl}(t) \text{ for } t \in [0,\infty[\tag{7}$$

Further investigations regarding the conditions on the relaxation and the creep tensors have been made by Gurtin & Sternberg [3] and Day [9].

All latin indices refer to Cartesian coordinates. Unless otherwise mentioned, the summation convention is used.

All strains are assumed to be small compared with unity and all displacements are assumed to be small compared with the dimensions of the structure.

3. A compatible type of finite elements.

3.1 Variational principles.

The functional, see fig. 1,

$$\pi_k = \frac{1}{2} \int_V [\sigma_{ij} * d\epsilon_{ij}](\underset{\sim}{x},t)dV - \int_V [\bar{F}_i * du_i](\underset{\sim}{x},t)dV$$
$$- \int_{A_\sigma} [\bar{T}_i * du_i](\underset{\sim}{x},t)dA \qquad (8)$$

given by Gurtin [6] is chosen as a starting point for the derivation of the finite element equations, corresponding to an element of the compatible type.

The additional conditions are

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \text{ in } V \qquad (9)$$

$$\sigma_{ij} = G_{ijkl} * d\epsilon_{kl} \quad \text{in } V \qquad (10)$$

$$u_i = \bar{u}_i \quad \text{on } A_u \qquad (11)$$

Gurtin [6] has shown that, under these conditions, the state S = ($\underset{\sim}{\epsilon}$, $\underset{\sim}{\sigma}$, $\underset{\sim}{u}$) is a solution to the given viscoelastic problem when

1. The variation of π_k vanishes with respect to S.

2. The variations obey the additional conditions.

The variation of a functional can be expressed as follows:

$$\delta\pi(S) = \frac{d}{d\alpha}\,\pi(S + \alpha\delta S)_{\alpha=0} \qquad (12)$$

for an arbitrary choice of δS.

The uniqueness of the solution is also shown by Gurtin [6]. The functional π_k is the viscoelastic analogy to the potential energy for perfectly elastic materials, though it should be noted that π_k does not represent any physical energy. Energi-expressions for viscoelastic materials are given by Bland [8].

3.2 Spatial discretization.

An approximation of the functional π_k is obtained by dividing the structure into discrete elements and expressing the independent functions by a finite number of generalized values and sets of interpolation functions.

In this case the only independent functions, the displacements u_i, are chosen in the form

$$\{u(\underset{\sim}{x},t)\} = [N(\underset{\sim}{x})]\{v(t)\} \tag{13}$$

where [N] is chosen independent of the time t in accordance with the assumptions mentioned above. The stress components and the strain components depend on the displacements through the additional conditions.

$$\{\varepsilon(t)\} = [B]\{v(t)\} \tag{14}$$

As [N] is time-independent, so is [B]

$$\{\sigma(t)\} = \int_{-\infty}^{t} [C(t-\tau)][B]d\{\varepsilon(\tau)\} \tag{15}$$

By using the approximate displacements and (A1) and (A8) of the appendix we obtain

$$\begin{aligned}\pi_K^D = \tfrac{1}{2} \int_V \int_{-\infty}^{\tau=t} \int_{-\infty}^{\tau'=t-\tau} d\{v(\tau)\}^T[B]^T[G(t-\tau-\tau')][B]d\{v(\tau')\}dV \\ - \int_V \int^{\tau=t} \{\bar{F}(t-\tau)\}^T [N]d\{v(\tau)\}dV \\ - \int_{A_\sigma} \int_{-\infty}^{\tau=t} \{\bar{T}(t-\tau)\}d\{v(\tau)\}dA\end{aligned} \tag{16}$$

By replacing π_K by the approximation π_K^D and using the variational principle, we get a set of equations to determine the approximate displacements.

The variational principle and (12) give

$$\begin{aligned}\delta\pi_K(S) = \tfrac{1}{2}\int_V \int_{-\infty}^{t} \int_{-\infty}^{t-\tau} d\{\delta v(\tau)\}^T[B]^T[G(t-\tau-\tau')][B]d\{v(\tau')\}dV \\ +\tfrac{1}{2}\int_V \int_{-\infty} \int_{-\infty}^{t-\tau} d\{v(\tau)\}^T[B]^T[G(t-\tau-\tau')][B]d\{\delta v(\tau')\}dV \\ - \int_V \int_{-\infty}^{t} \{\bar{F}(t-\tau)\}^T[N]d\{\delta v(\tau)\}dV \\ - \int_{A_\sigma} \int_{-\infty}^{t} \{\bar{T}(t-\tau)\}^T[N]d\{\delta v(\tau)\}dA\end{aligned} \tag{17}$$

As the argument of [G] is symmetrical with respect to τ and τ'

so are the first two expressions of (17), and this can thus be written in the form

$$\delta\pi_K^D = \int_{-\infty}^{t}\int_{-\infty}^{t-\tau} d\{\delta v(\tau)\}^T [K(t-\tau-\tau')]d\{v(\tau')\}$$
$$- \int_{-\infty}^{t} \{R(t-\tau)\}^T d\{\delta v(\tau)\} \qquad (18)$$

where $[K(t)] = \int_V [B]^T[G(t)][B]dV \qquad (19)$

and $\{R(t)\} = \int_V [N]^T\{\bar{F}(t)\}dV + \int_{A_\sigma} [N]^T\{\bar{T}(t)\}dA \qquad (20)$

The elements of $\{\delta v(\tau)\}$ are replaced one by one by the unit step function $h(t)$ while the rest of the elements are taken as zero. This choice of $\{\delta v\}$ allows some of the integrations with respect to time to be carried out analytically, see appendix (A6). Now we get a simple system of equations.

$$\int_{-\infty}^{t} [K(t-\tau)]d\{v(\tau)\} - \{R(t)\} = 0 \qquad (21)$$

3.3 Time-discretization

The time-dependent functions of eq. (21) are approximated by continuous linear functions. The changes of the slope are chosen at $t_1 \le t_2 \le .... \le t_N$. (21) can then be written in the form

$$\sum_{i=1}^{N} ([K(t_N-t_i)] + [K(t_N-t_{i-1})]\{\Delta v_i\} = 2\{R(t_N)\} \qquad (22)$$

By using more refined approximations for the integral, we obtain a better accuracy, but also a coupling of the equations at different time steps.

The N'th displacement increment is easily determined from (22), if the previous increments are known.

$$\{\Delta v_N\} = ([K(0)] + [K(t_N-t_{N-1})])^{-1}$$
$$(2\{R(t_N)\} - \sum_{i=1}^{N-1}([K(t_N-t_{i-1})]+[K(t_N-t_i)])\{\Delta v_i\} \qquad (23)$$

By replacing the displacement increments by the total displacements in (23) we get

$$\{v_N\} = ([K(0)] + [K(t_N - t_{N-1})])^{-1}$$
$$(2\{R(t_N)\} - \sum_{i=1}^{N-1}([K(t_N - t_{i-1})] - [K(t_N - t_{i+1})])\{v_i\}) \quad (24)$$

This system of equations is identical with the one derived by Taylor & Chang [5]. It is important to note that discontinuities with respect to time do not make the method fail. For $\Delta t_1 = 0$ the well-known equations for elastic materials are obtained.

$$\{\Delta v_1\} = [K(0)]^{-1}\{R(t_1)\} \quad (25)$$

3.4 Examples.

3.4.1 Elastic materials

For an elastic material the relaxation tensor and consequently the stiffness matrix are time-independent. (24) then becomes

$$[K]\{v(t)\} = \{R(t)\} \quad (26)$$

This set of equations will be recognized as similar to the equations derived by Zienkiewicz [7].

3.4.2 Constant strain velocity for isotropic materials.

The investigation is restricted to the case of constant stresses. In this case, assuming constant strain velocity, (1) becomes

$$\sigma_{ij} = G_{ijkl}(t)\,\epsilon_{kl}(0) + \int_0^t G_{ijkl}(\tau)d\tau\;\dot{\epsilon}_{kl}(0) \quad (27)$$

The relaxation tensor of an isotropic material can be expressed by

$$G_{ijkl}(t) = \frac{G_2(t) - G_1(t)}{3}\delta_{ij}\delta_{kl} + \frac{G_1(t)}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad (28)$$

By combining (27) and (29) and solving with respect to G_1 and G_2 we get

$$G_1(t) = G_1(0)\exp(-\alpha t)$$
$$G_2(t) = G_2(0)\exp(-\beta t) \quad (29)$$

where the dilatation $\epsilon = \frac{1}{3}\epsilon_{KK}$ (30)

and the deviations $e_{ij} = \epsilon_{ij} - \delta_{ij}\epsilon$ (31)

$\alpha = \dot{e}_{ij}/e_{ij} \qquad \beta = -\dot{\epsilon}/\epsilon$ (32)

i.e. the general form

$$G_{ijkl}(t) = \frac{1}{3}(G_2(0)\exp(-\alpha t) - G_1(0)\exp(-\beta t))\delta_{ij}\delta_{kl} + \frac{1}{2}G_1(0)\exp(-\beta t)(\delta_{ik}\delta_{lj} + \delta_{il}\delta_{jk}) \qquad (33)$$

is the relaxation tensor of a material subjected to the initial strains

$$\epsilon_{ij} = G^{-1}_{ijkl}\,\sigma_{kl} \qquad (34)$$

and the constant strain velocities

$$\dot{\epsilon}_{ij} = \beta(\epsilon_{ij} - \frac{1}{3}\,\delta_{ij}\,\epsilon_{kk}) + \frac{1}{3}\,\delta_{ij}\,\alpha\epsilon_{kk} \qquad (35)$$

at constant stress.

4. A hybrid type of viscoelastic finite element.

4.1 Variational principles.

Consider the functional given by Gurtin [6]

$$\pi_C = \tfrac{1}{2}\int_V [\epsilon_{ij} * d\sigma_{ij}](\underset{\sim}{x},t)dV - \int_{A_u} [T_i * du_i](\underset{\sim}{x},t)dV \tag{36}$$

with the additional conditions

$$\sigma_{ij,j} + \bar{F}_i = 0 \quad \text{in } V \tag{37}$$

$$\sigma_{ij} = \sigma_{ji} \quad \text{in } V \tag{38}$$

$$\sigma_{ij}n_j = \bar{T}_i \quad \text{in } A_\sigma \tag{39}$$

$$T_i^{(I)} + T_i^{(II)} = 0 \quad \text{on } A_i \tag{40}$$

$$\sigma_{ij}n_j = T_i \quad \text{on } A_i \tag{41}$$

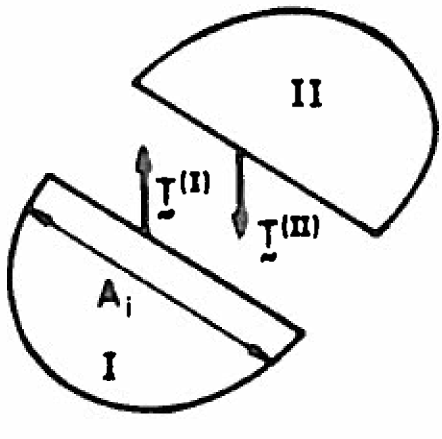

Fig 2

π_C is analogous to the complementary energy for perfectly elastic materials. It should be noted that π_C, like the functional π_K defined in (8), does not represent any physical energy. The additional conditions (39) and (40) are introduced in the functional π_C by the method of Lagrange multipliers $\lambda_i(\underset{\sim}{x},t)$. The method of introducing the equilibrium conditions along inner surfaces into the functional is treated by Pian & Tong [10].

$$\pi_H = \pi_C - \int_{A_i} (T_i^{(I)} + T_i^{(II)}) * d\lambda_i(\underset{\sim}{x}, t)dA$$
$$- \int_{A_\sigma} (\sigma_{ij}n_j - \bar{T}_i) * d\lambda_i(\underset{\sim}{x}, t)dA \qquad (42)$$

The variation of the modified functional π_H is found by means of (12).

$$\delta\pi_H = \delta\pi_C - \int_{A_i} (T_i^{(I)} + T_i^{(II)}) * d\delta\lambda_i(\underset{\sim}{x}, t)dA$$
$$- \int_{A_\sigma} (\sigma_{ij}n_j - \bar{T}_i) * d\delta\lambda_i(\underset{\sim}{x}, t)dA$$
$$- \int_{A_i} (\delta T_i^{(I)} + \delta T_i^{(II)}) * d\lambda_i(\underset{\sim}{x}, t)dA$$
$$- \int_{A_\sigma} \delta\sigma_{ij}n_j * d\lambda_i(\underset{\sim}{x}, t)dA \qquad (43)$$

The state S is determined by the independent functions: the stresses $\underset{\sim}{\sigma}$ and the Lagrange multipliers $\underset{\sim}{\lambda}$

$$S = (\underset{\sim}{\sigma}, \underset{\sim}{\lambda}) \qquad (44)$$

By choosing $\delta\underset{\sim}{\sigma} = 0$ and $\delta\underset{\sim}{\lambda} = n(t)\delta\underset{\sim}{\lambda}(\underset{\sim}{x})$ we get

$$\delta\pi_H = - \int_{A_i} (T_i^{(I)} + T_i^{(II)})\delta\lambda_i(\underset{\sim}{x})dA$$
$$- \int_{A_\sigma} (\sigma_{ij}n_j - \bar{T}_i)\delta\lambda_i(\underset{\sim}{x})dA \qquad (45)$$

As $\lambda_i\delta(\underset{\sim}{x})$ can be arbitrarily chosen, it follows that $\delta\pi_H = 0$ if $T_i^{(I)} + T_i^{(II)} = 0$ on A_i and $\sigma_{ij}n_j = \bar{T}_i$ on A_σ.

By choosing $d\underset{\sim}{\lambda} = 0$ and $\delta\underset{\sim}{\sigma} = h(t)\delta\underset{\sim}{\sigma}(\underset{\sim}{x})$ and applying the symmetry of J_{ijkl}.

$$\delta\pi_H = \int_V \epsilon_{ij}\delta\sigma_{ij}(\underset{\sim}{x})dV - \int_{A_u} \delta\sigma_{ij}n_j\bar{u}_i(\underset{\sim}{x})dA$$
$$- \int_{A_i} \delta\sigma_{ij}n_j\lambda_i(\underset{\sim}{x})dA - \int_{A_\sigma} \delta\sigma_{ij}n_j\lambda_i(\underset{\sim}{x})dA \qquad (46)$$

By means of the additional condition (38) and the Gauss transformation, we get

$$\delta\pi_H = \int_A \delta\sigma_{ij}n_ju_i(\underset{\sim}{x})dA - \int_{A_u} \delta\sigma_{ij}n_j\bar{u}_i(\underset{\sim}{x})dA$$
$$- \int_{A_i} \delta\sigma_{ij}n_j\lambda_i(\underset{\sim}{x})dA - \int_{A_\sigma} \delta\sigma_{ij}n_j\lambda_i(\underset{\sim}{x})dA \qquad (47)$$

Then the variational principle states that

$$\lambda_i = \bar{u}_i \quad \text{on } A_u$$

and $\lambda_i = u_i$ on A_i and A_σ

By dividing the structure into NEL elements we obtain

$$\begin{aligned}\pi_H = \sum_{I}^{NEL} \Big(\tfrac{1}{2} \int_{V_I} & (J_{ijkl} * d\sigma_{ij} * d\sigma_{kl})(\underset{\sim}{x},t)dV \\ & - \int_{A_I} (\sigma_{ij}n_j * du_i)(\underset{\sim}{x},t)dA \\ & + \int_{A_{\sigma_I}} (\bar{T}_i * du_i)(\underset{\sim}{x},t)dA \Big)\end{aligned} \tag{48}$$

which is the modified functional to be used in the derivation of the finite element equations.

4.2 Stress modes and spatial discretization

The structure is divided into NEL elements and the stresses of each element I are expressed by

$$\{\sigma(\underset{\sim}{x},t)\}_I = [P(x)]_I \{\beta(t)\}_I + \{\sigma 1(\underset{\sim}{x},t)\}_I \tag{49}$$

where

$[P]_I$ is made time-independent,

$\{\beta\}_I$ are the stress modes of the element and

$\{\sigma 1\}_I$ satisfies the inhomogeneous part of the equations of equilibrium in the I'th element.

$\{\sigma\}_I$ and $\{\sigma 1\}_I$ contain generalized stresses since the symmetry of the stress tensor is used. The displacements along the element surface are continuous and expressed by

$$\{u(\underset{\sim}{x},t)\} = [N(\underset{\sim}{x})]_I \{v(t)\}_I \tag{50}$$

$[N]_I$ is made time-independent, and

$\{v\}_I$ are the nodal displacements of the I'th element.

Because of the continuity of the displacements, the nodal displacement are not quite independent and a transformation is required.

$$\{v\}_I = [S]_I \{v\} \qquad (51)$$

where $\{v\}$ are the independent global displacements. By introducing the global displacements and stress modes we obtain the approximate functional.

$$\begin{aligned}\pi_H^D = \sum_I^{NEL} (\int_{V_I} \int_{-\infty}^{\tau=t} \int_{-\infty}^{T'=t-\tau} & f(\{\beta(\tau)\}_I^T [P]_I^T + \{\sigma 1(\tau)\}_I^T) \\ & [J(t-\tau-\tau')]_I d([P]_I \{s(\tau)\}_I + \{\sigma 1(\tau)\}_I) dV \\ - \int_{A_I} \int_{-\infty}^{\tau=t} & (\{\beta(t-\tau)\}_I^T [F]_I^T + \{T1(t-\tau)\}_I^T) [N]_I d\{v(\tau)\}_I dA \\ + \int_{A_{\sigma_I}} \int_{-\infty}^{\tau=t} & \{\bar{T}(t-\tau)\}_I^T [N]_I d\{v(\tau)\}_I dA\end{aligned} \qquad (52)$$

where [F] is defined by

$$\{T\}_I = [F]_I \{\beta\}_I \qquad (53)$$

Furthermore the following symbols are introduced

$$[H1(t)]_I = \int_{V_I} [P]_I^T [J(t)][P] dV \qquad (54)$$

$$[H2(t)]_I = \int_{V_I} [P]_I^T [J(t)]_I dV \qquad (55)$$

$$[H3(t,\tau,\tau')] = \int_{V_I} \{\sigma 1(\tau)\}_I [J(t-\tau-\tau')] \{\sigma 1(\tau')\}_I dV \qquad (56)$$

$$[G1]_I = \int_{A_I} [F]_I^T [N]_I dA \qquad (57)$$

$$\begin{aligned}[G2(t)]_I = & - \int_{A_I} \{T1(t)\}_I^T [N]_I dA \\ & + \int_{A_{\sigma_I}} \{\bar{T}(t)\}_I^T [N]_I dA\end{aligned} \qquad (58)$$

By means of the variational principle and (12), the variation of π_H^D is written

$$\begin{aligned}\delta\pi_H^D = & \frac{d}{d\alpha} \pi_H^D(\{v(t)\} + \alpha\{\delta v(t)\}, \\ & \sum_I^{NEL} (\{\beta(t)\}_I + \alpha\{\delta\beta(t)\}_I))\Big|_{\alpha=0} \\ = & \sum_I^{NEL} (\int_{-\infty}^{t} \int_{-\infty}^{t=\tau} d\{\delta\beta(\tau')\}_I^T [H1(t-\tau-\tau')]_I d\{\beta(\tau)\}_I)\end{aligned}$$

$$+ \int_{-\infty}^{t} \int_{-\infty}^{t-\tau} d\{\delta\beta(\tau')\}_I^T [H2(t-\tau-\tau')]_I d\{\sigma 1(\tau)\}_I$$
$$- \int_{-\infty}^{t} \{\beta(t)\}_I^T [G1]_I [S]_I d\{\delta v(\tau)\}$$
$$- \int_{-\infty}^{t} \{\delta\beta(t-\tau)\}_I [G1]_I [S]_I d\{v(\tau)]$$
$$- \int_{-\infty}^{t} [G2(t-\tau)]_I [S]_I d\{\delta v(\tau)\}) \qquad (59)$$

$\{\delta v(t)\}$ and $\{\delta\beta(t)\}_I$ may be chosen arbitrarily. One choice is $\{\delta v(t)\} \doteq 0$ and $\delta\beta_i(t) = h(t)$ successively while the rest of the stress parameters are zero. This choice and (59) give

$$\sum_{I}^{ELN} (\int_{-\infty}^{t} [H1(t-\tau)]_I d\{\beta(\tau)\}_I +$$
$$\int_{-\infty}^{t} [H2(t-\tau')]_I d\{\sigma 1(\tau')\}_I$$
$$- [G1]_I [S]_I \{v(t)\}) = 0 \qquad (60)$$

Another choice is $\{\delta\beta\}_I = 0$ and $\delta v_i(t) = h(t)$ while the rest of the global displacements are zero. With this choice we obtain

$$\sum_{I}^{ELN} ([S]_I^T [G1]_I^T \{\beta(t)\}_I + [S]_I^T [G2(t)]_I^T) = 0 \qquad (61)$$

4.3 Time discretization

It is assumed that like the compatible element, all functions are piecewise linear and continuous with respect to time t for $t \in [t_0, t_N]$ with discontinuities in the slope at $t_0 \le t_1 \le \ldots\ldots \le t_N$. Then (60) can be written

$$\sum_{I}^{NEL} ([H1(0)]_I + [H1(t_N - t_{N-1})]_I)\{\Delta\beta_N\}_I$$
$$= \sum_{I}^{NEL} (2[G1]_I [S]_I \{v_N\}$$
$$- \sum_{i}^{N-1} ([H1(t_N - t_i)]_I + [H1(t_N - t_{i-1})]_I)\{\Delta\beta_i\}$$
$$- \sum_{i}^{N} ([H2(t_N - t_i)]_I + [H2(t_N - t_{i-1})]_I)\{\Delta\sigma 1_i\}_I) \qquad (62)$$

As $\{\Delta\beta\}_I$ is independent from one element to another, $\{\Delta\beta_N\}_I$ can be found if $([H1(0)]_I + [H1(t_N - t_{N-1})])_I$ can be inverted. Assuming this and substituting $\{\beta_N\}_I = \{\beta_{N-1}\}_I + \{\Delta\beta_N\}$ and $\{\Delta\beta_N\}_I$

determined from (62) into (61), we obtain

$$2\sum_I^{NEL}\left([S]_I^T[G1]_I^T([H1(0)]_I+[H1(t_N - t_{N-1})]_I)^{-1}\right.$$

$$[G1]_I[S]_I\{v_N\}$$

$$=\sum_I^{NEL}\left([S]_I^T[G2(t_N)]_I-[S]_I^T[G1]_I^T\{\beta_{N-1}\}_I\right.$$

$$+[S]_I^T[G1]_I^T([H1(0)]_I+[H1(t_N - t_{N-1})]_I)^{-1}$$

$$\left(\sum_i^{N-1}(H1(t_N - t_i)]_I+[H1(t_N - t_{i-1})]_I)\{\Delta\beta_i\}_I\right.$$

$$\left.\left.+\sum_i^{NEL}(H2(t_N - t_i)]_I+[H2(t_N - t_{i-1})]_I\{\Delta\sigma 1_i\}_I)\right)\right) \qquad (63)$$

$\{v_N\}$ and $\{\Delta\beta_N\}_I$ may now be determined from (63) and (62).

Neglecting the inhomogeneous part of the equilibrium equations (62) and (63) reduce to

$$([H1(0)]_I + [H1(t_N - t_{N-1})]_I\{\Delta\beta_N\}_I$$

$$= 2[G1]_I[S]_I\{v_N\} -$$

$$\sum_i^{N-1}([H1(t_N - t_{i-1})]_I+[H1(t_N - t_i)]_I\{\Delta\beta_i\}_I \qquad (64)$$

$$2\sum_I^{NEL}[S]_I^T[G1]_I^T([H1(0)]_I+[H1(t_N - t_{N-1})]_I)^{-1}$$

$$[G1]_I[S]_I\{v_N\}$$

$$=\sum_I^{NEL}([S]_I^T[G2(t_N)]_I$$

$$+[S]_I^T[G1]_I^T([H(0)]_I+[H(t_N - t_{N-1})]_I)^{-1}$$

$$\sum_i^{N-1}([H1(t_N - t_i)]_I+[H1(t_N - t_{i-1})]_I)\{\Delta\beta_i\}_I) \qquad (65)$$

5. A special type of relaxation and creep tensors.

The numerical methods involved in the integration with respect to time demand a high computer capacity. This may be avoided in the case of certain type of material, whose relaxation tensor can be written

$$[G(t)] = [G_0] + \sum_{k=1}^{Q} [G_k]\exp(-t/T_k^G) \tag{66}$$

By applying (3) we obtain

$$[J(t)] = [J_o] + \sum_{k=1}^{Q} [J_k]\exp(-t/T_k^J) \tag{67}$$

This kind of material has been treated by Zak [11] in one-dimensional cases.

5.1 Zak's method applied to the compatible type.

(66) substituted into (19) gives

$$[K(t)] = \int_V [B]^T([G_o] + \sum_{k=1}^{Q} [G_k]\exp(-t/T_k^G)[B]dV$$

$$= [K_o] + \sum_{k=1}^{Q} [K_k]\exp(-t/T_k^G)$$

(23) may then be written in the form

$$(2[K_o] + \sum_{k=1}^{Q} [K_k](1+\exp(-(t_N - t_{N-1})/T_k^G))\{\Delta v_N\}$$

$$= 2\{R(t_N)\} - \sum_{i=1}^{N-1} 2[K_o]\{\Delta v_i\}$$

$$- \sum_{k=1}^{Q} [K_k] \sum_{i=1}^{N-1} ((\exp((-t_N - t_i)/T_k^G)+\exp(-(t_N - t_{i-1})/T_k^G))\{\Delta v_i\}$$

$$= 2\{R(t_N)\} - 2[K_o]\{v_{N-1}\} - \sum_{k=1}^{Q} [K_k]\{\alpha_{N,k}\} \tag{68}$$

where $\{v_{N-1}\} = \sum_{i}^{N-1} \{\Delta v_i\}$, (69)

$$\{\alpha_{N,k}\} = \exp(-\Delta t_N/T_K^G)(1 + \exp(-\Delta t_{N-1}/T_K^G))\{\Delta v_{N-1}\}$$

$$+ \exp(-\Delta t_N/T_K^G)\{\alpha_{N-1,k}\} \tag{70}$$

and $\Delta t_N = t_N - t_{N-1}$ (71)

The displacements are determined from (68) and the recursion formula (70).

5.2 Zak's method applied to hybrid elements satisfying the homogeneous equilibrium equations.

Substituting (67) into (54) gives

$$[H1(t)]_I = \int_{V_I} [P]_I^T [J_o][P]_I dV + \sum_k^Q \int_{V_I} [J_k][P]_I dV \exp(-t/T_{kI}^J)$$
$$= [H1_o]_I + \sum_{k=1}^Q [H1_k]_I \exp(-t/T_{kI}^J) \qquad (72)$$

Substituting this into (64) gives

$$\sum_{i=1}^N (2[H1_o]_I + \sum_{k=1}^Q [H1_k]_I (\exp(-(t_N - t_i)/T_{kI}^J))$$
$$+ \exp(-(t_N - t_{i-1})/T_{KI}^J))\{\Delta\beta_i\}_I$$
$$= 2[G1]_I[S]_I\{v_N\} \qquad (73)$$

(73) is rewritten

$$(2[H1_o]_I + \sum_{k=1}^Q [H1_k]_I (1+\exp(-\Delta t_N/T_{kI}^J))\{\beta_N\}_I$$
$$= 2[G1]_I[S]_I\{v_N\} +$$
$$\sum_{k=1}^Q [H1_k]_I ((1 + \exp(-\Delta t_N/T_{kI}^J))\{\beta_{N-1}\}_I - \{\alpha_{N,k}\}_I) \qquad (74)$$

where

$$\{\alpha_{N,k}\}_I = \exp(-\Delta t_N/T_{kI}^J)(1 + \exp(-\Delta t_{N-1}/T_{kI}^J)).$$
$$\{\Delta\beta_{N-1}\}_I$$
$$+ \exp(-\Delta t_N/T_{kI}^J)\{\alpha_{N-1,k}\}_I \qquad (75)$$

and

$$\{\Delta\beta_N\}_I = \{\beta_N\}_I - \{\beta_{N-1}\}_I \qquad (76)$$

$\{\beta_N\}_I$ may be determined from (74), and substituted into (61). This gives

$$\sum_I^{NEL} -[S]_I^T[G1]_I^T(2[H1_o]_I + \sum_{k=1}^Q [H1_k]_I (1 + \exp(-\Delta t_N/T_{kI}^J))^{-1}.$$

$$(2[G1]_I [S]_I^T \{v_N\} + \sum_{k=1}^{Q} [H_k]_I (1 + \exp(-\Delta t_N/T_{kI}^J))\{\beta_{N-1}\}_I$$

$$- \sum_{k=1}^{Q} [H_k]_I \{\alpha_{N,k}\}_I + \{R\}_I) = 0 \tag{77}$$

This is rewritten in the form

$$\sum_{I}^{ELN} 2[S]_I^T [G1]_I^T (2[H1_o]_I + \sum_{k=1}^{Q} [H1_k]_I (1 + \exp(-\Delta t_N/T_{kI}^J)))^{-1}$$

$$[G1]_I [S]_I \{v_N\}$$

$$= \{R\} - \sum_{I}^{ELN} [S]_I^T [G1]_I^T (2[H1_o]_I + \sum_{k=1}^{Q} [H1_k]_I (1 + \exp(-\Delta t_N/T_{kI}^J))^{-1}$$

$$\sum_{k=1}^{Q} [H1_k]_I ((1 + \exp(-\Delta t_N/T_{kI}^J))\{\beta_{N-1}\}_I - \{\alpha_{N,k}\}_I) \tag{78}$$

The nodal displacements can be determined from (78).

Bibliography

1. Christensen, R.M. [1971]
Theory of viscoelasticity
Academic Press.

2. Rabotnov [1969]
Creep problems in structural members
North-Holland.

3. Gurtin, M.E. & Sternberg, E. [1962]
Arch. f. rat. mechs. and anal. 11 291-356

4. White, J.L. [1968]
Finite elements in linear viscoelasticity
Conf. (2d): matrix meth. struct. mechs.

5. Taylor, R.L. & T.Y. Chang. [1966]
An approximate method for thermo-viscoelastic stress analysis.
Nuclear eng. & des. 4 (1966) 21-28

6. Gurtin, M.E. [1963]
Variational principles in the linear theory of viscoelasticity
Arch.f. rat. mechs. & anal. 13 179-191

7. Zienkiewicz, O.C. [1972]
The finite element method in engineering science.
McGraw-Hill

8. Bland, D.R. [1960]
The theory of linear viscoelasticity
Pergamon Press

9. Day, W.A. [1971]
Time-reversal and symmetry of the relaxation function of a linear viscoelastic material.
Arch. f. rat. mechs. & anal. 40, (1971), 155-158

10. Pian, T.H.H. & P. Tong. [1969]
Basis of finite element methods for solid continua
Int. J. for numerical meth. in eng. 1, (1969), 3-28

11. Zak, A.R. [1968]
Structural analysis of realistic solid-propellant materials.
J. of spacecraft 5, 1968, 270-275

12. Wissmann, W. [1969]
Rechnerische Erfassung des Kriechens und Schwindens bei Stahlbeton, Spannbeton und Verbundtragwerken.
Beton- und Stahlbetonbau 11, 1969, 258-263

13. Aagaard Sørensen, K. & Bent Højlund Rasmussen. [1971]
Redistribution of stresses due to shrinkage and creep in uncracked composite beams.
Bygningsstatiske medd. 3-4, 1971.

Appendix

The Stieltjes' convolution product is defined by

$$\varphi * d\psi = \int_{-\infty}^{t} \varphi(t-\tau)d\psi(\tau); \quad t > 0 \tag{A1}$$

where $\varphi(t) = \psi(t) = 0; \; t \in \,]-\infty, 0[$

φ is continuous, $t \in [0,\infty[$

ψ is piecewise continuous $t \in [0,\infty[$

Furthermore

$\omega(t) = 0 \qquad t \in \,]-\infty,0[$

and $\omega(t)$ is continuous $t \in [0,\infty[$

The following rules of calculation are observed:

$$\varphi * d\psi = \psi * d\varphi \tag{A2}$$

$$\varphi * d(\psi * d\omega) = (\varphi * d\psi) * d\omega = \varphi * d\omega * d\psi \tag{A3}$$

$$\varphi * d(\psi + \omega) = \varphi * d\psi + \varphi * d\omega \tag{A4}$$

$$\varphi * d\psi = 0 \Rightarrow \varphi \equiv 0 \vee \psi \equiv 0 \tag{A5}$$

$$\varphi * dh = \varphi \tag{A6}$$

$$h = \begin{cases} 1 & t \in \,]-\infty, 0[\\ 0 & t \in [0, \infty[\end{cases} \tag{A7}$$

(A3) may be written

$$\varphi * d\omega * d\psi = \int_0^{\tau=t} \int_0^{\tau'=t-\tau} \psi(t-\tau-\tau')d\omega(\tau')d\psi(\tau) \tag{A8}$$

The case of ψ differentiable in $]0,\infty[$ turns (A2) into

$$\varphi * d\psi = \varphi(t)\psi(0) + \int_0^t \varphi(t-\tau)\,\frac{d\psi(\tau)}{d\tau}\,d\tau \tag{A9}$$

SUMMARY

The present paper introduces a class of hybrid linearly viscoelastic finite elements, additionally a compatible class is developed. The equations are derived by variational principles applied to a complementary functional modified by Lagrange multipliers as well as a potential energy functional. The timedependence introduced by Stieltjes' convolution product thus allowing discontinuities in stresses, strains, displacements and loads. To avoid time-consuming numerical integrations the class of creep and relaxation functions is limited to constants and exponential functions in time.

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REPORTS

R 1.	Askegaard, Vagn and P. Thoft-Christensen: Spændingsoptiske lag og tøjningsmålere. 1967.	Out of print
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