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DANMARKS TEKNISKE HØJSKOLE

STRUCTURAL RESEARCH LABORATORY

TECHNICAL UNIVERSITY OF DENMARK

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OPTIMUM DESIGN OF REINFORCED CONCRETE SHELLS AND SLABS

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ABSTRACT

In this paper a method is presented for optimization of the reinforcement required to resist arbitrary internal forces in reinforced concrete shells. As the design principles suggested are applicable to shells of arbitrary shape, including slabs, subjected to all types of loading, the method is of a general nature. The procedure also indicates whether the thickness and strength of the concrete shell are sufficient to ensure a statically admissible stress field in the shell. The internal forces include normal and shear forces and bending and twisting moments. The design criteria correspond to the ultimate limit state.

SYNOPSIS

The paper deals with the optimum design of reinforced concrete shells of arbitrary shape subjected to an arbitrary load. The shape of the shell is assumed to be given. The investigation thus applies to shells, plates, folded plates, and slabs. The thickness of the shell, which may be nonuniform, is assumed to be given.

The load of the shell is assumed to be given.

In the general case, the reinforcement is assumed to consist of one or two parallel layers of orthogonal reinforcing net. The directions of the bars are referred to as the x- and y-directions. In case of two layers, these directions will be assumed to be common for the layers.

The position of the reinforcement relative to the middle surface of the shell is assumed to be given. This position is usually governed by the requirements to concrete cover.

The internal forces in any section of the shell are assumed to be known. These include the normal forces N_{xo} and N_{yo}, the shear force N_{xyo}, the bending moments M_x and M_y, and the twisting moments M_{xy}.

The method is an ultimate limit design method. The bearing capacity of the shell has thus been adopted as the basic design criterion. The problem of buckling of the shell or slab is not discussed in this paper.

The tensile strength of the concrete is neglected. The normal sections of the concrete are assumed to be cracked except where compressive stresses occur.

The concrete stresses are assumed to be uniformly distributed in the compression zone.

Compressive reinforcement has not been considered. The method can easily be extended to include this, but in that case, the problem of buckling of the compressed reinforcing bars will have to be taken into account.

The design strengths, σ_a in the reinforcement and σ_b in the concrete, are assumed to be specified.

The problem to be solved is as follows:

1. Check if the thicknesss and strength of the concrete are sufficient at all points of the shell.
2. Calculate the minimum necessary cross-sectional areas of the reinforcement at any point of the shell surface. In the general case, this covers four quantities corresponding to two layers with two directions in each layer.

If the corresponding reinforcement is provided, it will be possible to indicate a statically admissible stress field in the shell.

The problem is of a general character. The method was first adopted for cylindrical shell caissons for Hanstholm Harbor, Denmark (Fig. 1), and later applied to similar structures in Africa and the United Kingdom.

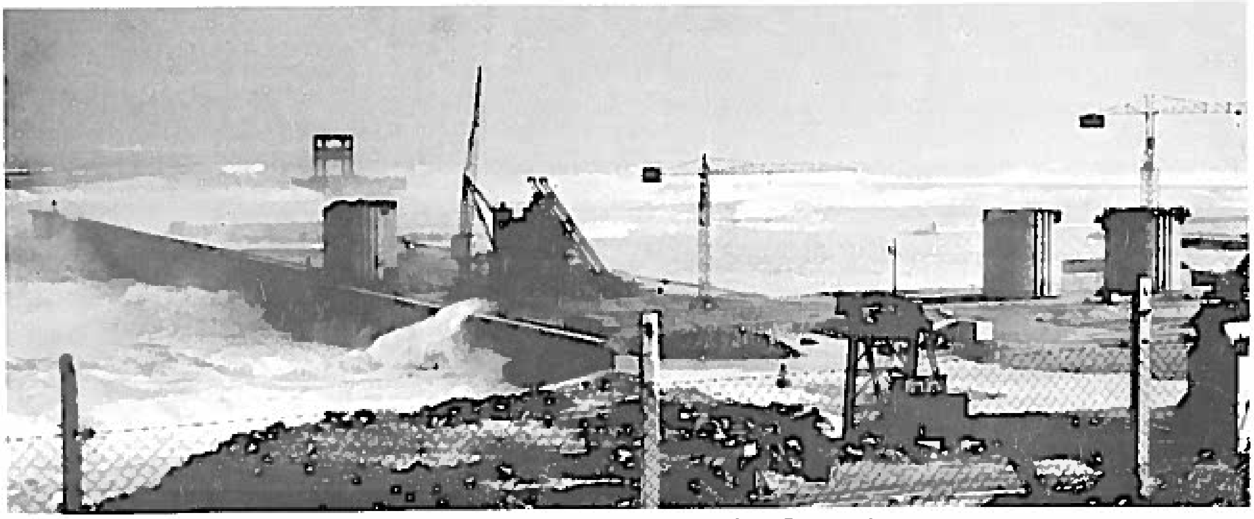


Fig. 1 Caissons in Hanstholm Harbor, Denmark

SANDWICH SHELL MODEL

In Fig. 2 an infinitesimal element of the shell has been isolated between consecutive sections parallel to the x- and y-directions. The lengths dx and dy of the sides of this element are taken as equal to unity.

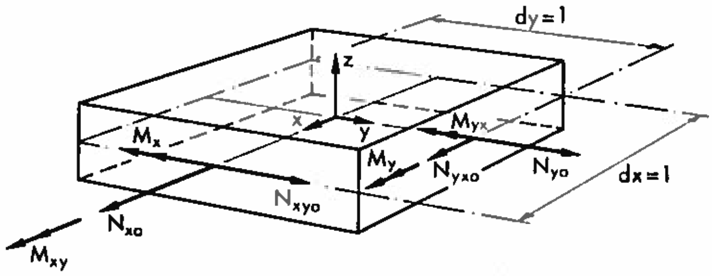


Fig. 2 Internal Forces

All normal forces and normal stresses are taken as positive corresponding to tension. In Fig. 2 the internal forces and moments per unit length of the shell are indicated. The orthogonal x- and y-axes are located in the middle surface of the shell, and the z-axis is perpendicular to this surface. For the present analysis, a corresponding sandwich element is substituted for the shell element proper. This element is shown in Fig. 3(a-d). The sandwich element consists of three layers. If their thicknesses are given, the geometry of the sandwich shell element is known. The question of the geometry of the sandwich shell will be discussed in a later section of this paper. For the present discussion, the geometry is assumed to be known. All forces and moments in the shell element (Fig. 2) may thus be resolved into membrane forces located at the middle surfaces of the top and bottom sandwich element, as illustrated in Fig. 3(a-d).

LIMIT ANALYSIS OF MEMBRANE SHELLS

A limit analysis of each of the outer sandwich layers of the shell (Fig. 3) may be carried out as suggested in [1].

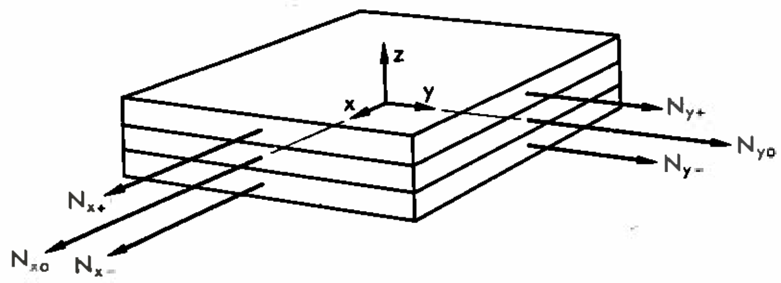


Fig. 3a Normal Forces Resolved into Membrane Forces in Sandwich Layers

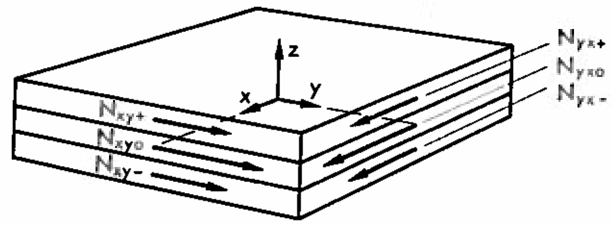


Fig. 3b Shear Forces Resolved into Membrane Forces in Sandwich Layers

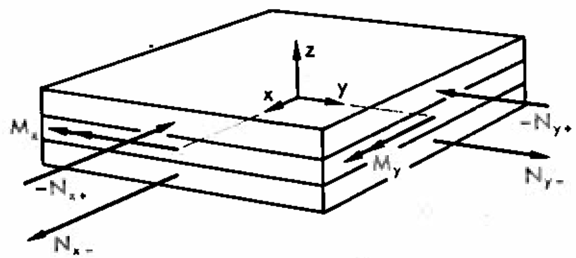


Fig. 3c Bending Moments Resolved into Membrane Forces in Sandwich Layers

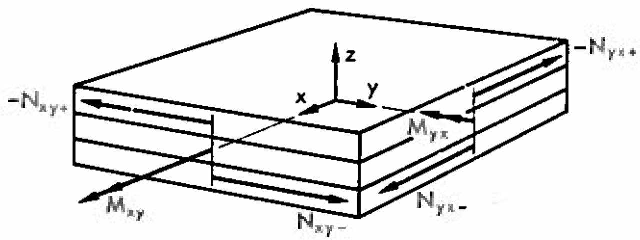


Fig. 3d Twisting Moments Resolved into Membrane Forces in Sandwich Layers

In Fig. 4, an element of a membrane shell has been isolated. The two sides are parallel to the orthogonal x- and y-directions. The third side represents a crack in the shell. The length of this side has been taken as equal to unity. The angle between the crack and the x-axis is denoted v. The inclination of the crack indicated in Fig. 4 corresponds to $N_{xy}\geq 0$. In Fig. 5, the corresponding situation is shown for $N_{xy}\leq 0$.

N_{xa} and N_{ya} denote the forces in the reinforcement in the x- and y-directions per unit length of these directions, respectively.

In both cases, equilibrium in the x- and y-directions requires

$$N_{xa} = N_x + |N_{xy}| \cot v \quad (1)$$

$$N_{ya} = N_y + |N_{xy}| \tan v \quad (2)$$

If both N_{xa} and N_{ya} are positive, the necessary reinforcement is proportional to $N_{xa} + N_{ya}$. Minimum of reinforcement thus corresponds to $v = 45^o$.

Consequently

$$N_{xa} = N_x + |N_{xy}| \quad (3)$$

$$N_{ya} = N_y + |N_{xy}| \quad (4)$$

Equations (3) and (4) are only valid as long as both N_{xa} and N_{ya} are positive. This requires

$$\left.\begin{matrix} N_x \\ N_y \end{matrix}\right\} \geq -|N_{xy}|$$

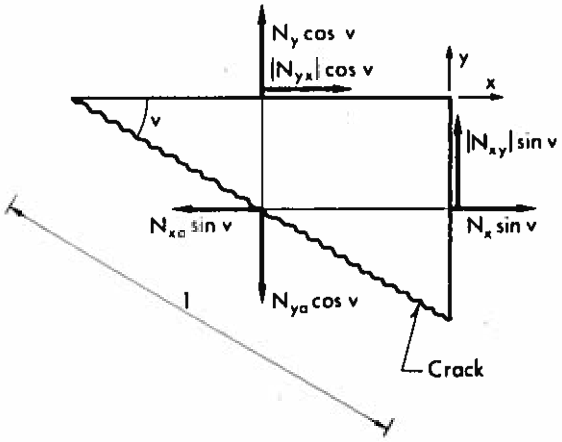


Fig. 4 Case 1: Reinforcement Required in Both Directions. $N_{xy}>0$

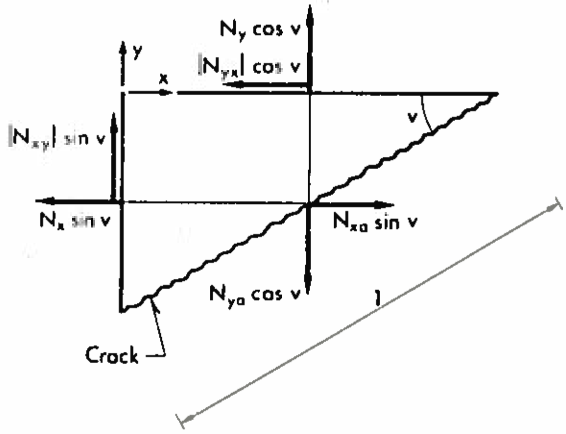


Fig. 5 Case 1: Reinforcement Required in Both Directions. $N_{xy}<0$

The above situation ($v = 45^o$) is illustrated in Fig. 6. The principal, compressive membrane force N_b in the concrete occurs in sections perpendicular to the crack, for instance, in the section along the line of symmetry of Fig. 6. The part below this line has been isolated in Fig. 7. Equilibrium of this element requires

$$N_b = -2|N_{xy}| \quad (5)$$

As all normal forces are taken as positive corresponding to tension, N_b is negative.

Equations (3) through (5) are no longer valid when N_{xa} is negative, i.e., when $N_x<-|N_{xy}|$. This situation is illustrated in Fig. 8 which corresponds to the situation $N_x<N_y$. Reinforcement is only required in the y-direction. Equilibrium in the x- and y-directions requires

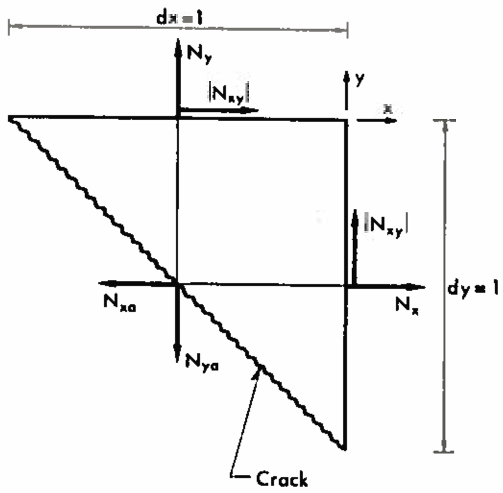


Fig. 6 Case 1: Optimum Inclination of Crack

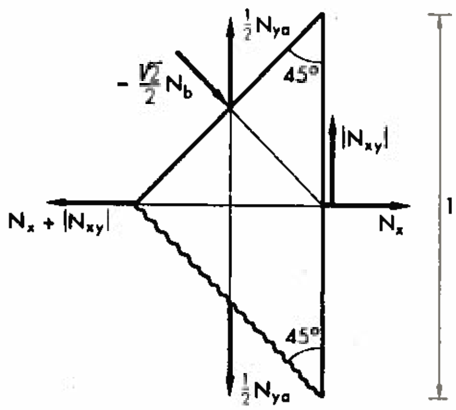


Fig. 7 Case 1: Principal, Compressive Membrane Force, N_b

$$\tan v = \frac{|N_{xy}|}{|N_x|} \qquad (6)$$

$$N_{ya} = N_y + |N_{xy}| \tan v \qquad (7)$$

Equations (6) and (7)

$$\boxed{N_{ya} = N_y - \frac{N_{xy}^2}{N_x}} \qquad (8)$$

Equation (8) is valid for $N_{ya} \geq 0$, i.e. for $N_x N_y \leq N_{xy}^2$.

The principal, compressive membrane force in the concrete occurs in sections perpendicular to the crack, for instance in the section through the origin. The part of the element below this section is isolated in Fig. 9. Equilibrium of the element shown in Fig. 8 requires that the resultant of the two forces acting on the side along the y-axis pass through the common intersection point of the three remaining forces. The resultant thus forms the angle v with the x-axis, as shown in Figs. 8 and 9. Equilibrium of the element in Fig. 9 requires

$$-N_b = |N_x| \sec^2 v = |N_x| (1+\tan^2 v) \qquad (9)$$

Equations (6) and (9)

$$\boxed{N_b = N_x + \frac{N_{xy}^2}{N_x}} \qquad (10)$$

When N_{ya} is negative, a corresponding situation occurs. This is illustrated in Fig. 10,

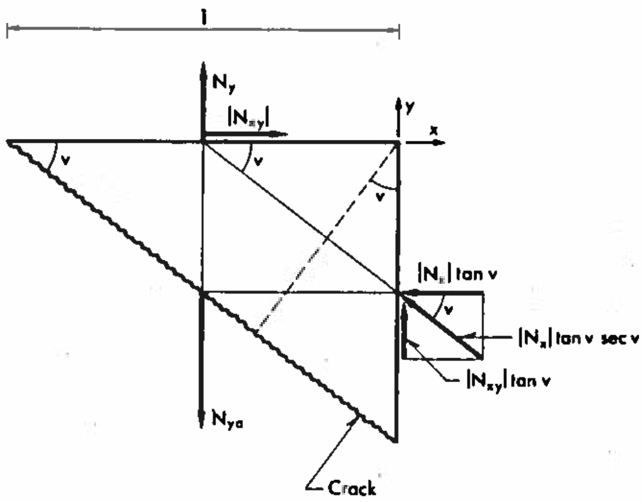


Fig. 8 Case 2: Reinforcement Only Required in the y-Direction

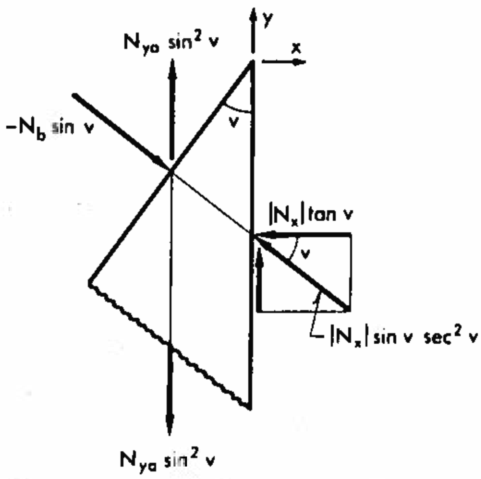


Fig. 9 Case 2: Principal, Compressive Membrane Force, N_b

corresponding to the case $N_y < N_x$. In this case reinforcement is only required in the x-direction. Equilibrium of the element in the y- and x-directions requires

$$\cot v = \frac{|N_{xy}|}{|N_y|} \qquad (11)$$

$$N_{xa} = N_x + |N_{xy}| \cot v \qquad (12)$$

Equations (11) and (12)

$$\boxed{N_{xa} = N_x - \frac{N_{xy}^2}{N_y}} \qquad (13)$$

Equation (13) is only valid for positive values of N_{xa}, i.e., for $N_x N_y \leq N_{xy}^2$.

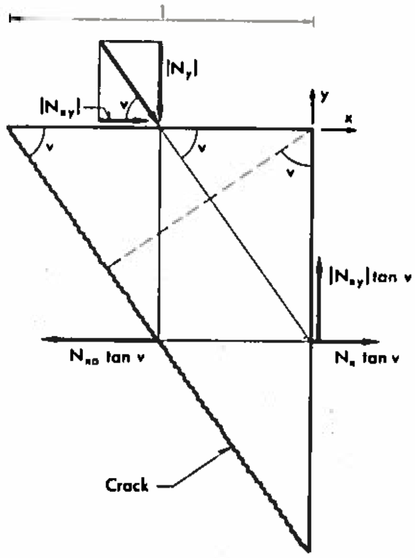


Fig. 10 Case 3: Reinforcement Only Required in the x-Direction

In Fig. 11 the upper part of the element in Fig. 10 above the line through the origin and perpendicular to the crack has been isolated. Equilibrium of this element requires

$$-N_b = |N_y| \operatorname{cosec}^2 v = |N_y| (1+\cot^2 v) \quad (14)$$

Equations (14) and (11)

$$\boxed{N_b = N_y + \frac{N_{xy}^2}{N_y}} \qquad (15)$$

If both N_x and N_y are negative and $N_x N_y \geq N_{xy}^2$, no reinforcement is required. In this case, the principal membrane force in the concrete corresponding to maximum compression can be calculated from the conventional formula

$$\boxed{N_b = ½(N_x+N_y) - ½\sqrt{(N_x-N_y)^2+4N_{xy}^2}} \qquad (16)$$

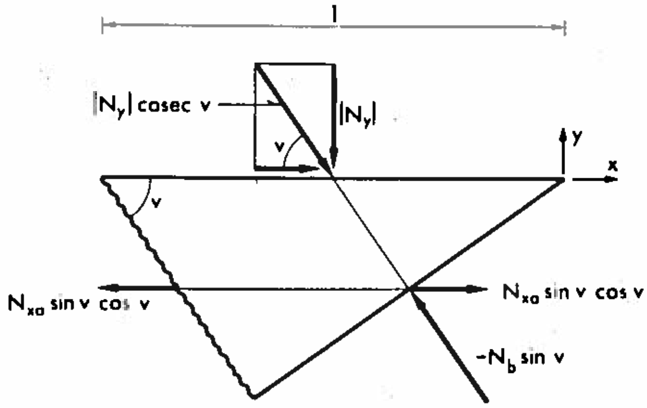


Fig. 11 Case 3: Principal Compressive Membrane Force, N_b

The four cases discussed above are summarized in the diagram in Fig. 12. Each combination of N_x, N_y and N_{xy} represents a point in this diagram, and the formulae indicated in the corresponding sector of the diagram apply to the case in question. The directions in which reinforcement is required are indicated by the directions of the hatching in the sectors.

GEOMETRY OF SANDWICH SHELL

The concrete is assumed to be cracked as illustrated in Fig. 13. The principal, compressive membrane force N_b in the concrete is assumed to be resisted by uniformly distributed stresses σ_b. This requires that the depth c of the neutral axes be

$$\boxed{c = \frac{N_b}{\sigma_b}} \qquad (17)$$

If the only internal forces in the section are the normal force N and the bending moment M as illustrated in Fig. 13, the moment of these forces with regard to the tensile reinforcement is

$$M_a = M - Ne \qquad (18)$$

where e denotes the eccentricity of the tensile reinforcement.

The moment M_a has to be resisted by the concrete compressive stresses. Consequently

$$M_a = -c\sigma_b(h - \tfrac{1}{2}c) \qquad (19)$$

where h denotes the effective depth of the reinforcement.

With the notation

$$\mu = -\frac{M_a}{h^2\sigma_b} \qquad (20)$$

Eq. (19) leads to

$$c = h(1 - \sqrt{1 - 2\mu}) \qquad (21)$$

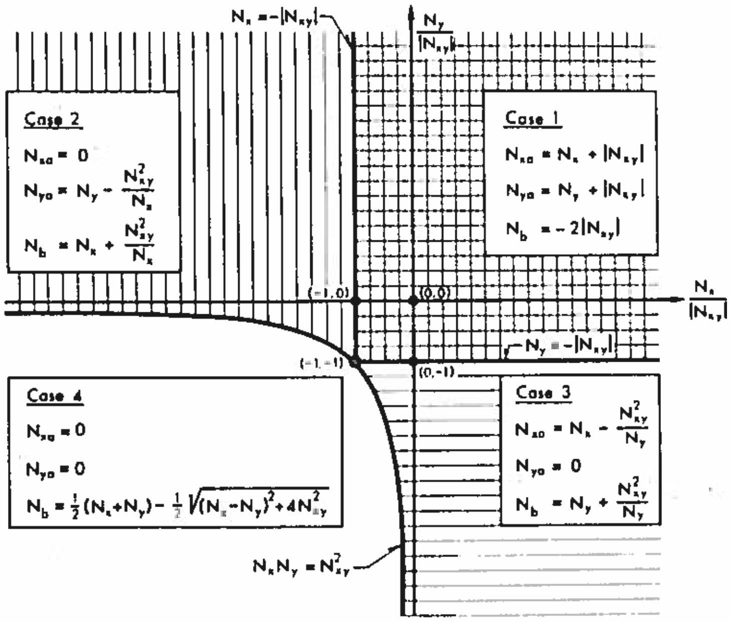


Fig. 12 Chart of Relevant Design Formulae for Any Combination of Membrane Forces

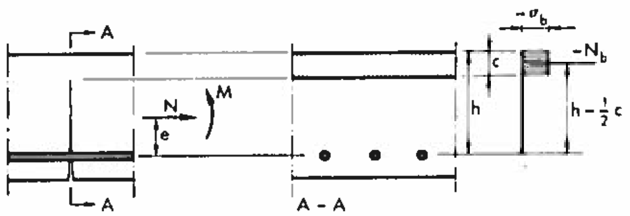


Fig. 13 Cracked, Reinforced Concrete Section

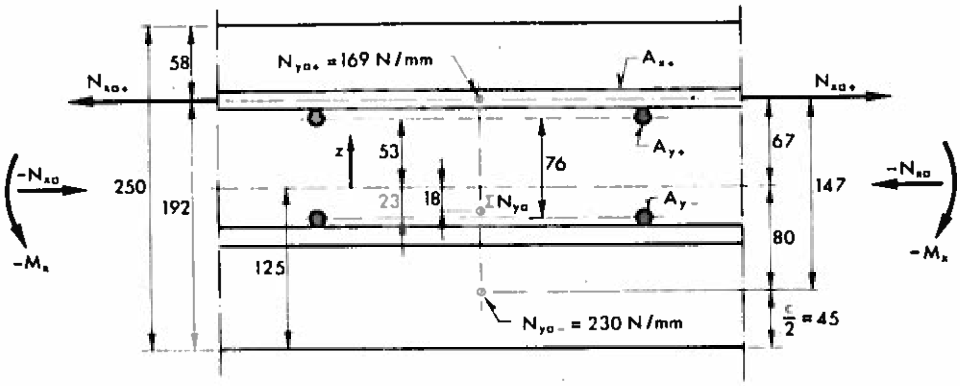
According to Eqs. (5), (10), and (15), N_{xy} contributes to N_b and thus affects the required value of c according to Eq. (17). On the other hand, N_{xy} cannot be calculated until the geometry of the shell has been defined, and this geometry depends upon c. The problem can be solved by trial and adjustment. In most cases, it appears practical to base the first estimate on the predominant bending moment and Eqs. (18), (20), and (21), thus neglecting the effect of N_{xy}. If N_{xy} is not zero, the value of c thus found will have to be increased.

The middle surfaces of the outer sandwich layers thus correspond to the centroid of the reinforcement resisting the predominant bending moment and to the centroid of the corresponding concrete compression zone, respectively. The method is illustrated in the numerical example in the next section. The geometry of the sandwich shell is thus governed by the predominant bending moment. If this, for instance, is M_x, a certain force N_{ya} according to Eqs. (4) and (8) may have to be resisted at the middle surface of one or both of the outer sandwich layers. However, the centroids of the reinforcement in the y-direction do not usually coincide with these middle surfaces. For this reason the resultant ΣN_{ya} of the two N_{ya} forces (N_{ya+} and N_{ya-}) has to be resolved into forces at the centroids of the corresponding reinforcement. This will also be illustrated in the numerical example in the next section. If the resultant ΣN_{ya} is not located between the centroids of the reinforcement in the y-direction, reinforcement in this direction will only be required in one of the layers. The resultant can then be resisted by this reinforcement in connection with a compressive force in one of the outer sandwich layers.

NUMERICAL EXAMPLE

N_{xo} = -120 N/mm
N_{yo} = 300 N/mm
N_{xyo} = 170 N/mm
M_x = -83,000 N
M_y = 12,000 N
M_{xy} = 800 N

σ_b = -7 N/mm^2
σ_a = 270 N/mm^2
Thickness of shell: 250 mm



All dimensions are in millimetres unless otherwise stated

Fig. 14 Position of Reinforcement and Membrane Forces

M_x is the predominant bending moment. Effective depth of corresponding reinforcement (Fig. 14)

$$h_x = 192 \text{ mm}$$

M_a is found by substituting $-M_x$ for M and N_x for N in Eq. (18)

$$M_a = 83{,}000 + 120 \cdot 67 = 91{,}000 \text{ N}$$

Equation (20)

$$\mu = -\frac{91{,}000}{192^2(-7)} = 0.35$$

Equation (21)

$$c = 192\,(1 - \sqrt{1 - 2 \cdot 0.35}\,) = 87 \text{ mm}$$

Both N_{xyo} and M_{xy} contribute to N_{xy-} and thus to N_b according to Eq. (10). Consequently, the necessary depth c of the compression zone must be expected to be somewhat more than 87 mm. Estimated value

$$c = 90 \text{ mm}$$

The middle surfaces of the corresponding outer sandwich layers are then located at z = 67 mm and z = -80 mm, respectively (Fig. 14).

By resolving the internal forces as illustrated in Fig. 3 the following membrane forces are found at the bottom sandwich layer

$$N_{x-} = -\frac{91{,}000}{147} = -619 \text{ N/mm}$$

$$N_{y-} = 300\frac{67}{147} + \frac{12{,}000}{147} = 137 + 82 = 219 \text{ N/mm}$$

$$N_{xy-} = 170\frac{67}{147} + \frac{800}{147} = 77 + 5 = 82 \text{ N/mm}$$

As $N_{x-} < -|N_{xy-}|$, no reinforcement is required in the x-direction.

As $N_x < N_y$, Eqs. (8) and (10) apply (Case 2 in Fig. 12)

$$N_{ya-} = 219 + \frac{82^2}{619} = 219 + 11 = 230 \text{ N/mm}$$

$$N_b = -619 - 11 = -630 \text{ N/mm}$$

Equation (17)

$$c = \frac{-630}{-7} = 90 \text{ mm}$$

No correction of the estimated value of c is thus required.

Membrane forces in the top sandwich layer

$$N_{x+} = 619 - 120 = 499 \text{ N/mm}$$

$$N_{y+} = (300 - 137) - 82 = 81 \text{ N/mm}$$

$$N_{xy+} = (170 - 77) - 5 = 88 \text{ N/mm}$$

This corresponds to Case 1 in Fig. 12.

Equations (3) to (5)

$$N_{xa+} = 499 + 88 = 587 \text{ N/mm}$$

$$N_{ya+} = 81 + 88 = 169 \text{ N/mm}$$

$$N_b = -2 \cdot 88 = -176 \text{ N/mm}$$

This principal membrane force is located 58 mm below the top surface of the shell (Fig. 14). If it is resisted by uniform concrete stresses in a compression zone, the depth of this zone will be 2·58 = 116 mm and the stress

$$\sigma = -\frac{176}{116} = -2 \text{ N/mm}^2$$

The compressive stress is thus much below the design value.

Required cross-sectional area of reinforcement in the x-direction per unit length of the y-direction

In the top layer:

$$A_{x+} = \frac{587}{270} = 2.17 \text{ mm}$$

In the bottom layer:

No reinforcement required.

The resultant of N_{ya+} and N_{ya-} is

$$\Sigma N_{ya} = 169 + 230 = 399 \text{ N/mm}$$

It is located at

$$z = \frac{169 \cdot 67 - 230 \cdot 80}{399} = -18 \text{ mm}$$

The corresponding layers of reinforcement in the y-direction are located at $z = +53$ mm and $z = -23$ mm, respectively.

Required cross-sectional areas of reinforcement in these layers per unit length of the x-direction are

$$A_{y+} = \frac{399}{270}\,\frac{23 - 18}{53 + 23} = 0.10 \text{ mm}$$

$$A_{y-} = \frac{399}{270}\,\frac{53 + 18}{53 + 23} = 1.38 \text{ mm}$$

NOTATION

x, y, z	Rectilinear coordinates (Fig. 2)
$N, N_{xo}, N_{yo}, N_{xyo}, N_{yxo}$	Normal and shear forces per unit length of shell section (Fig. 2)
$M, M_x, M_y, M_{xy}, M_{yx}$	Bending and twisting moments per unit length of shell section (Fig. 2)
N_x, N_y, N_{xy}, N_{yx}	Normal and shear forces per unit length of outer layers of sandwich shell (Fig. 3). Additional suffices + and - refer to top and bottom layers.
N_{xa}, N_{ya}	Forces in reinforcement per unit length of section (Fig. 4)
v	Angle between crack and x-axis (Figs. 4 and 5)
N_b	Principal, compressive membrane force per unit length of compression zone. As all normal forces are taken as positive corresponding to tension, N_b is negative.
e	Eccentricity of tensile reinforcement (Fig. 13)
h, h_x, h_y	Effective depths of tensile reinforcement (Fig. 13)
c	Depth of rectangular stress block (Fig. 13)
M_a	Moment with respect to tensile reinforcement (Eq. (19))
μ	Non-dimensional parameter (Eq. (20))
σ	Stress

σ_a, σ_b	Design strengths of steel and concrete
A_x, A_y	Required cross-sectional areas of reinforcement (per unit length) in the x- and y-directions. Additional suffices + and - refer to top and bottom layers.

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