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THE PHOTO-ELASTIC EFFECT IN THREE-DIMENSIONAL STATES OF STRESS

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The photo-elastic effect in three-dimensional states of stress.

Preface

The report has been prepared on the basis of a study of the literature.

It is the aim of the report to clarify the basic principles governing the photo-elastic conditions in three-dimensional stress fields and to describe and evaluate the existing methods of calculating the state of polarization along a light ray passing an optically anisotropic and heterogeneous body.

A linear relationship between stresses, strain and photo-elastic effect is assumed throughout.

Contents

In photo-elastic investigations of inhomogeneous, three-dimensional stress fields, an optical signal is obtained, which depends on the stress field along the light-wave's path through the model.

As interpretation of this signal is complicated, it is usually found preferable to undertake a mechanical slicing of the model (freezing method). The desire to avoid this destructive treatment leads to the question of how much information on the stress field along the path of the light can be derived from this integrated optical signal.

The question, an answer to which is attempted in the conclusion of this report, requires knowledge of the methods available for calculation of the polarization field along the light path when the stress field is known. This problem is dealt with in the report.

It is solely the secondary principal stresses in sections perpendicular to the direction of propagation of the light that determine alterations in the state of polarization. The relationship between the primary and the secondary principal stresses is described in most textbooks on photo-elasticity [12], [29].

The report gives a description of the assumptions and principles for existing methods of describing a light-wave's passage of an optically heterogeneous and anisotropic body. The terms heterogeneous and anisotropic refer only to the optical properties of the body. From a mechanical point of view we assume that the body is homogeneous and isotropic, so that Hooke's law is valid throughout, with constant modolus of elasticity and Poisson's ratio.

The report differentiates between kinematic*) descriptions, in which a relationship is assumed between the refractive index (Fresnel's ellipsoid) and stresses or strains, and dynamic*) descriptions, in which the assumptions are the electromagnetic field theory and an assumption on the dielectric tensor's dependence on the strain or stress tensor.

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*) The terms kinematics and dynamics are taken from mechanics. Kinematics is the description of the movement of the bodies - here, the change in state of polarization - without reference to the forces causing the movement. Dynamics is the study of the movement of bodies and of the forces - here, electrical and magnetic fields - causing the movement.

1. Basic description of polarized light

A plane light-wave, which is a superposition of electromagnetic wave-trains in which the individual electrical vibrations have the same direction of propagation, can be characterized by means of the following three properties:

a) the light-wave can be monochromatic, when the individual vibrations have the same frequency, i.e. same colour.

b) the light-wave can be polarized, which is to say that the electromagnetic field vector is subjected to a controlled movement, or unpolarized. Light consisting of a superposition of polarized and unpolarized light is partially polarized.

c) a monochromatic light-wave can be coherent, i.e. the individual vibrations have the same phase at the same time and place; light which is not coherent is called incoherent.

As the following aims at presenting the light-waves in a manner that lends itself to a description of photo-elastic phenomena, it will suffice to treat plane and monochromatic, polarized light-waves. It is not necessary to use coherent light. Interference also occurs when the interfering light-waves originate from the same light source. As this light exhibits the same photo-elastic effect as a single vibration, the following account is further restricted to a single electromagnetic oscillation.

1.1 The light-ellipse, Poincaré's sphere

A polarized light-wave moving in the x-direction can be described by means of the components of the electrical vector in two directions x and y at right-angles to each other in a plane perpendicular to the x- axis.

The symbols are explained in the notation.

$$x = a_1 \cos\Omega t$$
$$y = a_2 \cos(\Omega t + \delta^*) \qquad 1.1$$

where the origin is chosen such that $x = a_1$ at the time $t = 0$.
Eliminating t (see page A1), we get:

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} - 2\frac{xy}{a_1 a_2}\cos\delta^* = \sin^2\delta^* \qquad 1.2$$

i.e., the electrical vector describes an ellipse with the semi-axes

$$\left.\begin{matrix} d_1 \\ d_2 \end{matrix}\right\} = \frac{a}{\sqrt{2}} \sqrt{1 \pm \sqrt{1 - \sin^2 2\alpha \, \sin^2 \delta^*}} \qquad 1.3$$

*)In some presentations, however, the direction of the light is used as the basis for the terms left-handed or right-handed in which we have introduced

$$a_1 = a \cos\alpha$$
$$a_2 = a \sin\alpha \qquad 1.4$$

The axes of the ellipse form an angle with the x-axis, where we get by reduction

$$tg2\beta = \frac{2a_1 a_2}{a_1^2 - a_2^2} \cdot \cos\delta^* = tg2\alpha \cos\delta^* \qquad 1.5$$

The proof of 1.3 and 1.5 is given on pages A1-A2.

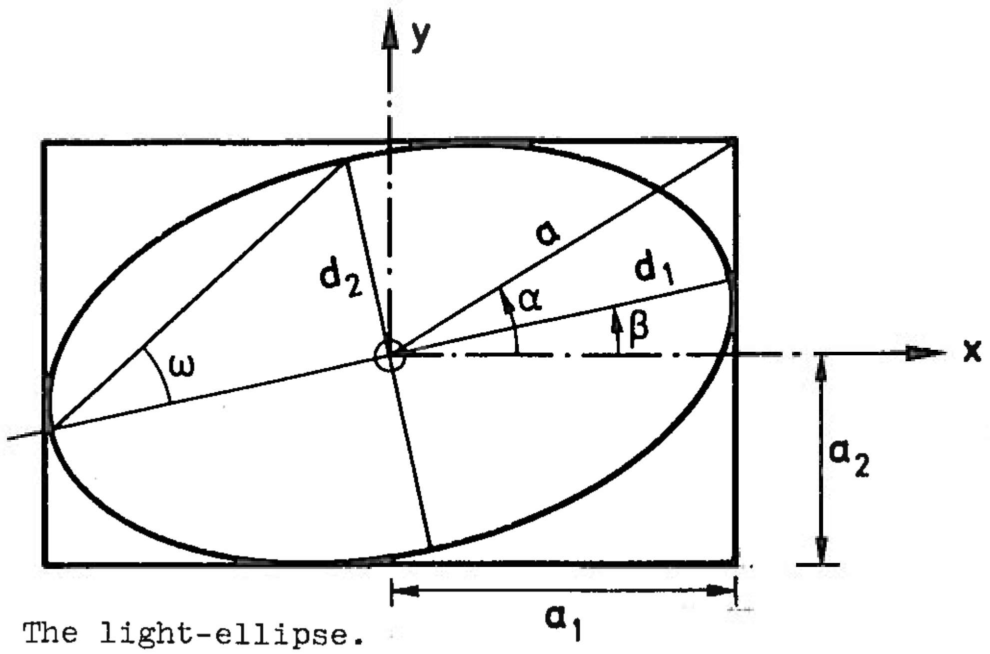


Fig. 1.1 The light-ellipse.

When the ellipse develops into a line or a circle, we have the two special cases:

a) Linear polarization; for the phase difference $\delta^* = 0$.
b) Circular polarization; for $\delta^* = \pm \frac{\pi}{2}$, the electrical field vector describes a circle. For $\delta^* = + \frac{\pi}{2}$, the vector moves in a clockwise direction, and the light is said to be left-handed. With this sign definition , we are facing the light.

It will be seen from 1.1 that the light-ellipse can be characterized by the three quantities a_1, a_2 and δ^* or α, β and the ellipticity ω, where ω has been introduced in

$$tg\omega = \frac{d_2}{d_1} \qquad 1.6$$

The light intensity that is proportional to a^2 is often of no interest, and we therefore put $a = 1$.

If we mark a point P with the coordinates (x, y, z) on a spherical surface with radius $a = 1$, such that

$$x = \cos 2\omega \cos 2\beta$$
$$y = \cos 2\omega \cos 2\beta$$
$$z = \sin 2\omega \qquad 1.7$$

the state of polarization will be described by the point P. The spherical surface is called Poincaré's sphere, see fig. 1.2 [1].

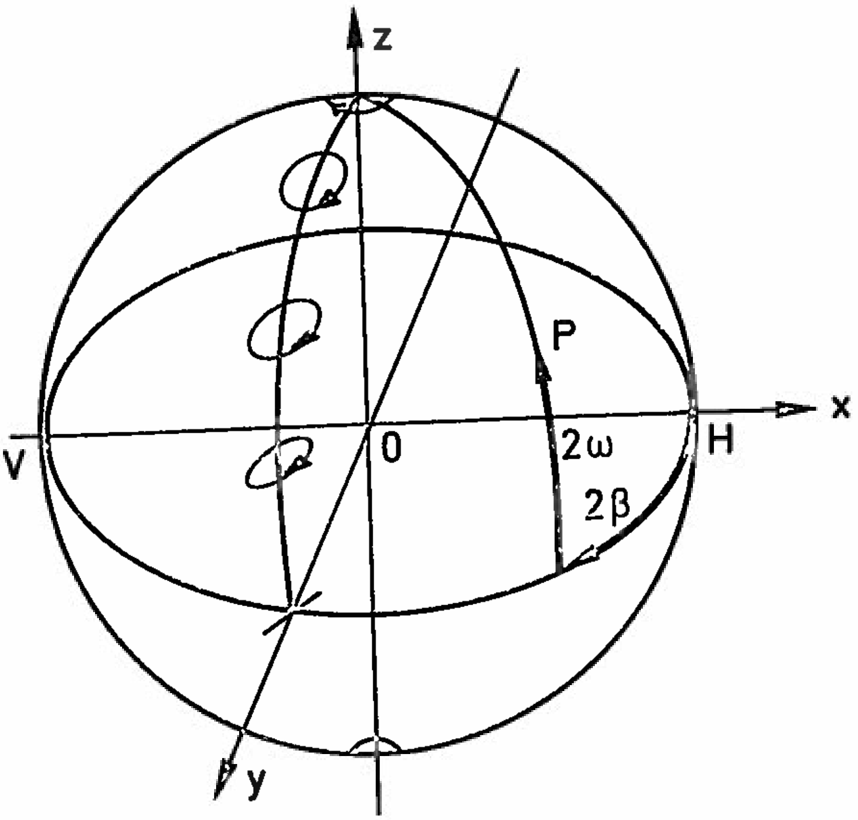


Fig. 1.2. Poincaré's sphere.

At the equator we have linearly polarized light, horizontal at H and vertical at V, when we take the x-axis as horizontal. At the upper and lower poles we have clockwise and anti-clockwise, circular polarization of the light, respectively. Over the rest of the sphere, we have elliptical polarization.

In order to facilitate a study of the conditions, light-ellipses can be drawn on the sphere, see fig. 1.2, in which the observer is imagined to be standing outside the sphere and observing the light-ellipse for a ray with direction from the centre.

Poincaré's sphere is principally used to determine the transformation from one polarization field into another during passage of a birefringent layer, see section 2.2.1.

1.2 Stokes' vector, Jones' vector

Instead of describing the state of polarization by α, β and ω, we can use Stokes' vector [2], [3]:

$$\bar{V} = \begin{bmatrix} S_o \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} a_1^2 + a_2^2 \\ a_1^2 - a_2^2 \\ 2a_1 a_2 \cos\delta^* \\ 2a_1 a_2 \sin\delta^* \end{bmatrix} \qquad 1.8$$

Here, there are then only three independent quantities as

$$S_o^2 = S_1^2 + S_2^2 + S_3^2 \qquad 1.9$$

The constituent parameters are the coordinates to Poincaré's sphere, since

$$\begin{bmatrix} S_o \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} a^2 \\ a^2\cos 2\omega \ \sin 2\beta \\ a^2\cos 2\omega \ \sin 2\beta \\ a^2\sin 2\omega \end{bmatrix} \qquad 1.10$$

see page A2.

Stokes' vector was originally proposed for a description of partially polarized light; in such case (1.9) does not apply, and in (1.8) the time averages of the amplitudes must be introduced.

In the present account, Stokes' vector is of particular interest because it is suitable for an analytical approach - Mueller calculations, see section 2.2.2 - to the transformation from one type of polarization into another.

In 1941 Jones proposed representing the state of polarization by a vector with two complex elements [4]:

$$\bar{D} = \begin{bmatrix} a_1 e^{i\Omega t} \\ a_2 e^{i(\Omega t-\delta)} \end{bmatrix} \qquad (1.11a)$$

The real part of the elements is the instantaneous value of the u- and v- components of the electrical vector in a coordinate system with axes parallel to the secondary principal directions.

In a coordinate system with fixed axes x and y, where the direction of propagation is the z-axis, Jones' vector takes the form

$$\bar{D}^{*} = \begin{bmatrix} a_1 e^{i\Omega t} \\ a_2 e^{i(\Omega t-\delta^{*})} \end{bmatrix} \qquad (1.11b)$$

where a_1 and a_2 are the amplitudes in this system.

Unlike Stokes' vector, Jones' vector can be used to describe interference vectors in coherent light. Jones' vector is more compact and it gives information on the instantaneous phase of the light-wave. On the other hand, it cannot describe partially polarized light. None of the above-mentioned properties is decisive for the photo-elastic phenomena.

2. Kinematic descriptions of a light-wave's passage through a birefringent layer with given characteristic properties.

Even though the first attempts at producing artificial birefringence by loading were carried out in 1816 by Brewster, Bio and Fresnel, not until 1841 do we find, in Neumann [5], a systematic investigation of the theory of photo-elasticity.

Neumann propounded the photo-elastic law for a three-dimensional strain field, and derived a differential equation for determining the state of polarization for a light-wave for the case in which the stress field varies along the direction of the light.

Since then, Poincaré, Coker and Filon, Mueller, Jones and other have obtained results equivalent to Neumann's differential equation. With this, it is, in principle, possible determine the state of polarization when the stress field is known along the light-wave.

However, only in special cases is it possible to arrive at the solution without the use of EDP.

It is particularly the inverse problem: determination of the stresses in a known polarization field, that is of interest. For this, the reader is referred to the conclusion of this report.

2.1 The continuous descriptions of Neumann and of Coker and Filon

On the assumption of small strains and small changes in velocity, and the assumption that the photo-elastic is solely dependent on the strains, Neumann deduced the following relationship between the velocity of the light in the loaded body and the principal strains:

$$\begin{aligned} v_a - v_o &= k_1\varepsilon_1 + k_2(\varepsilon_2 + \varepsilon_3) \\ v_b - v_o &= k_1\varepsilon_2 + k_2(\varepsilon_1 + \varepsilon_3) \\ v_c - v_o &= k_1\varepsilon_3 + k_2(\varepsilon_1 + \varepsilon_2) \end{aligned} \qquad (2.1)$$

where

v_o is the light velocity in the unloaded body.

v_a is the light velocity for a light-wave polarized in such a way that the electrical vector vibrates parallel with the direction of the first principal strain. The light-wave can thus propagate in a arbitrary direction in the plane set out by the directions of the second and third principal strains.

v_b and v_c are analogous to v_a.

ε_1, ε_2 and ε_3 are the principal strains.

k_1 and k_2 are strain-optical constants.

For a light-wave with arbitrary direction of propagation, the velocities are determined by means of Fresnel's ellipsoid. On the same assumption that the photo-elastic effect is only dependent on the strains, Neumann concluded that the strain ellipsoid and Fresnel's ellipsoid had the same principal axes, and he introduced the concept secondary principal strains and found a general relationship between these and the optical signal. A detailed account of Neumann's reflections are contained in [5] and [12].

Later, Maxwell [6] formulated a law corresponding to (2.1), but with the photo-elastic effect related to the stresses. In the linear-elastic zone - the only one in which we are working - the laws are identical.

Besides propounding a photo-elastic law, Neumann formulated a differential equation for the passage of the light through the birefringent body. Neumann assumed that both components of the light-wave had the same direction of propagation. This is an approximation due to the varying refractive indices. Besides this, only simple vectorial considerations are necessary.

The light-wave is considered as a superposition of two vibrations u and v in the secondary principal directions in the layer just passed by the wave, see fig. 2.1.

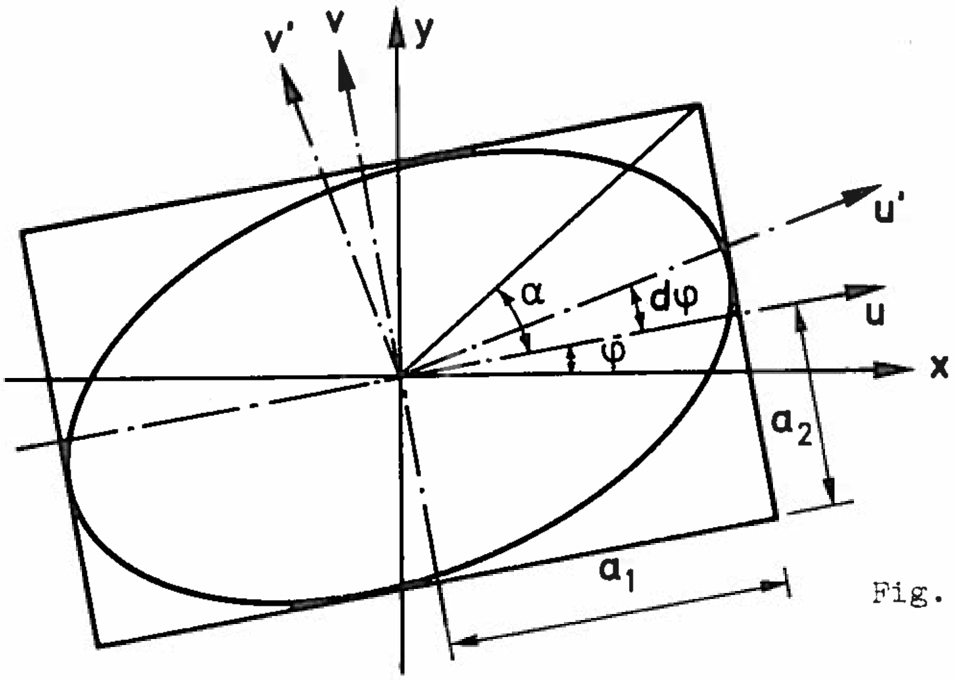


Fig. 2.1.

$$u = a_1 \cos(\Omega t + \delta_1)$$
$$v = a_2 \cos(\Omega t + \delta_2) \qquad (2.2)$$

If the light-wave is allowed to pass a new layer of thickness dz, where the secondary principal directions have undergone a rotation dφ, the state of polarization after the passage of the light can be described by the following system of equations:

$$\left.\begin{aligned} da_1 &= a_2 \cos\delta \, d\varphi \\ da_2 &= - a_1 \cos\delta \, d\varphi \end{aligned}\right\} \qquad (2.3)$$

$$d\delta = d\Delta + 2\cot 2\alpha \sin\delta \, d\varphi \qquad (2.4)$$

where

$d\delta$ is the retardation originating in the strain difference over the length dz.

α is determined from $tg\alpha = a_2/a_1$

$\delta = \delta_2 - \delta_1$

Neumann's derivation is reproduced on page A4-A5, with the above notation.

2.2 Discontinuous descriptions

In this section we will consider the effect of one or more birefringent layers with given properties.

The individual layers are of finite thickness, and the directions of the secondary principal stresses are assumed to be constant within each layer.

A single layer constitutes a linear retarder [27], while several layers form an elliptical retarder [27].

When a polarized light-wave passes a retarder, we achieve the change in the state of polarization by vectorial projection of the vibrations of the light-wave on two directions at right-angles to each other and letting one of the components undergo a phase displacement in relation to the other. In the linear retarder these directions are fixed and are identical with the principal directions, i.e. as in a plane photo-elastic model. In the elliptical retarder, these directions turn; we get a rotating effect.

A continuous description can be obtained from the discontinuous descriptions by making the layer thickness $\Delta z \rightarrow 0$, just as the reverse is possible.

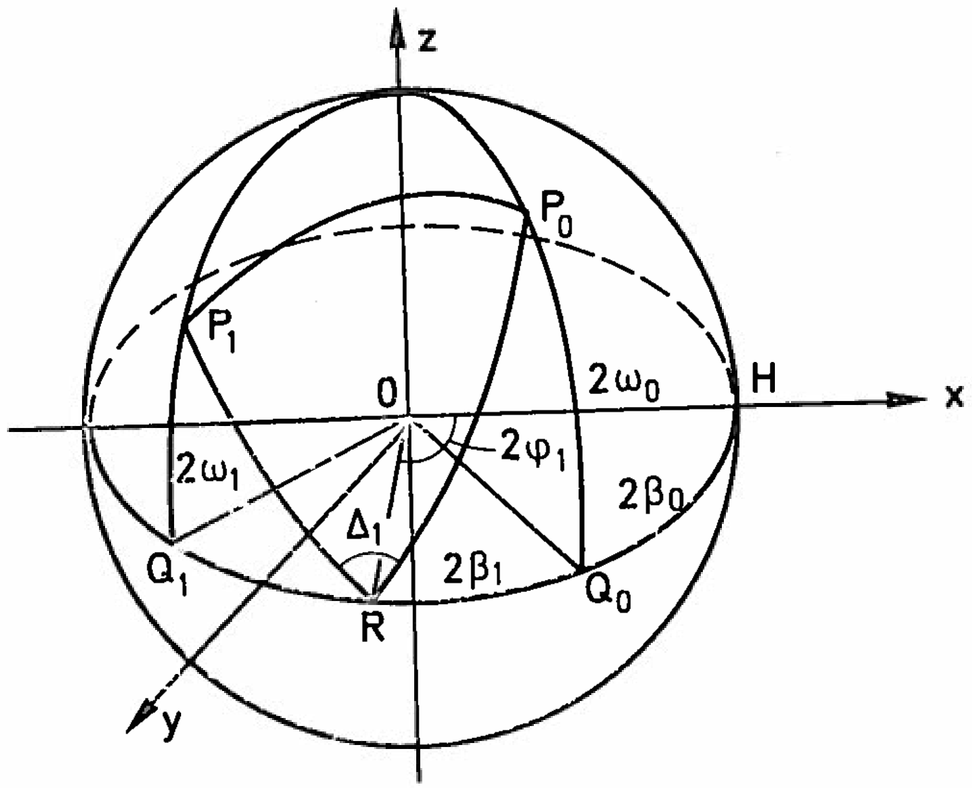
In this way the different descriptions can be derived from each other [8], [13]. Before proceeding, let us recapitulate the formulation of the problem in this chapter:

We wish to determine the state of polarization after passage of the layer on the basis of:

1. the state of polarization before passage of the layer, and
2. the retardation and orientation of the layer.

2.2.1 Geometrical descriptions

Poincaré's sphere is described in section 1.1. The transformation from one state of polarization into another during passage of the layer is obtained, as shown in fig. 2.2, by setting the orientation φ_1 of the retarder (the angle from the x-axis to the fast axis) at point R on the equator, so that $\angle HOR = 2\varphi_1$.



$\angle HOQ_o = 2\beta_o$ $\angle Q_oOP_o = 2\omega_o$	}	Light-ellipse for incident wave.
$\angle HOR = 2\varphi_1$ $\angle P_oRP_1 = \Delta_1$	}	Orientation and retardation of layer No. 1.
$\angle HOQ_1 = 2\beta_1$ $\angle Q_1OP_1 = 2\omega_1$	}	Light-ellipse for emerging wave.

Fig. 2.2. Poincaré's sphere.

The geometrical determination of the state of polarization after the passage is now carried out by means of a compass, the point being placed at R and the lead at P_o; an arc P_oP_1 is then traced in an anticlockwise direction such that

$< P_oRP_1 = \Delta_1$ = the retardation of the layer.

After the passage, the light-ellipse is characterized by point P_1 on the sphere:

$$P_1 : \begin{bmatrix} \cos 2\omega_1 \cos 2\beta_1 \\ \cos 2\omega_1 \sin 2\beta_1 \\ \sin 2\omega_1 \end{bmatrix}$$

where

$$\sin 2\omega_1 = \sin 2\omega_o \cos\Delta_1 + \sin 2(\varphi_1 - \beta_o)\cos 2\omega_o \sin\Delta_1$$

$$\mathrm{tg}\,2(\beta_1 - \varphi_1) = \frac{\sin 2\omega_o \sin\Delta_1 - \sin 2(\varphi_1 - \beta_o)\cos 2\omega_o \cos\Delta_1}{\cos 2\omega_o \cos 2(\varphi_1 - \beta_o)}$$

$$\mathrm{tg}\,2\beta_1 = \frac{\cos 2\omega_o[(1-\cos\Delta_1)\cos 2\varphi_1 \sin 2(\varphi_1 - \beta_o) + \sin 2\beta_o] + \sin 2\omega_o \sin\Delta_1 \cos 2\varphi_1}{\cos 2\omega_o[-(1-\cos\Delta_1)\sin 2\varphi_1 \sin 2(\varphi_1 - \beta_o) + \cos 2\beta_o] - \sin 2\omega_o \sin\Delta_1 \cos 2\varphi_1} \qquad (2.5)$$

The demonstration of this is given on page A6-A8.

After passage of n layers, the state of polarization will be characterized by point P_n. The transformation from P_1 into P_n can be constructed in the same way as that from P_1 to P_2, except that the point of the compass must be placed at a point that does not normally lie on the equator. This transformation is therefore characterized by three quantities.

By projection of Poincaré's sphere on a plane, we get the plane description (j-circle methods). Menges [8], who described the scattered-light method in 1940, independently of Weller, also introduced the j-circle method, which is a parallel projection on the plane through the equator of the sphere. Knowledge of this method has been disseminated particularly by Kuske [10], [11], and others.

We will not follow Kuske's procedure, but will explain the j-circle directly by means of Poincaré's sphere, see fig. 2.3 and 2.2, where the same notation is used. The j-vector gives the shape and orientation of the light-ellipse, the point of the vector corresponding to the projection of point P_i on Poincaré's sphere on the equatorial plane.

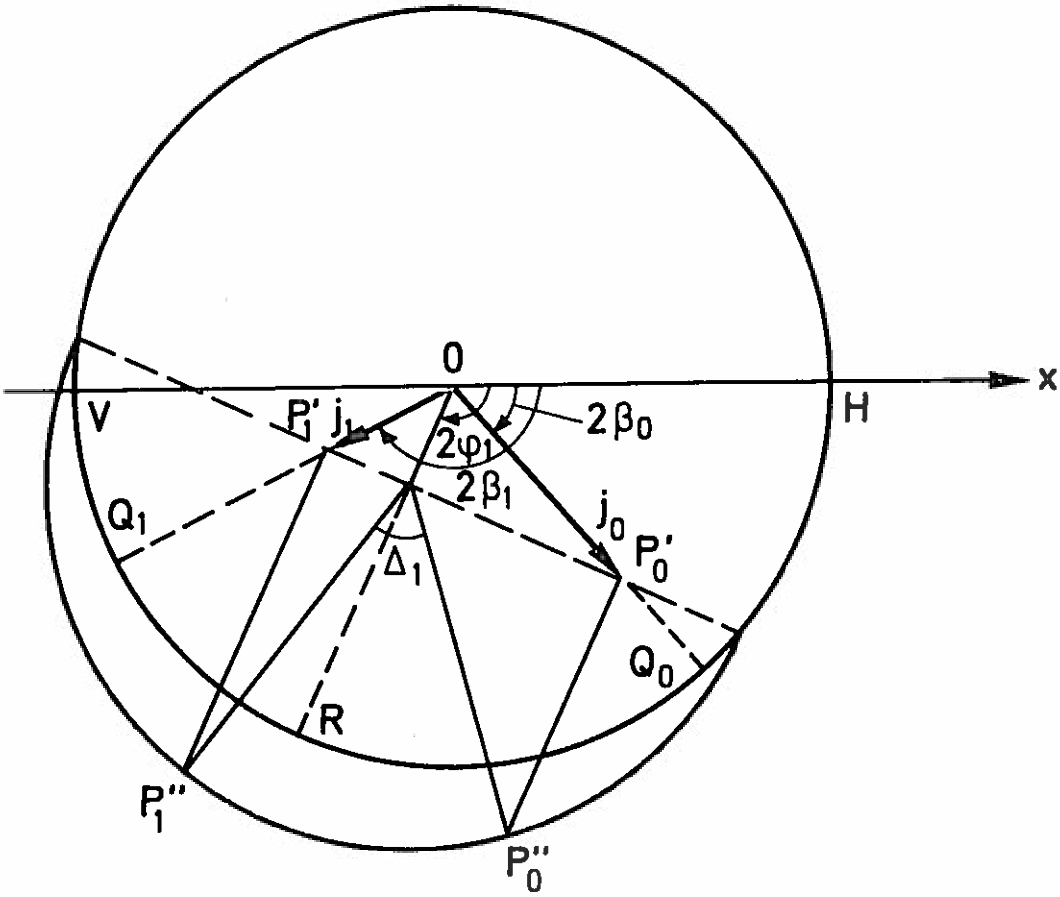


Fig. 2.3. The j-circle.

$$j_i = \cos 2\omega_i \qquad \angle HOP_i = 2\beta_i \qquad (2.6)$$

Menges measured the light-ellipse at different points along a ray through the model and marked out the j-vectors as shown in fig. 2.3. The points of the vectors form a curve, from which the orientation φ and the retardation Δ can easily be constructed. Kuske used the method to describe the state of polarization particularly in plates and shells, and proved the existence of the characteristic directions, which correspond to the isoclines in a plane stress field, see page 17. As a graphical method, the j-circle is more suitable than Poincaré's sphere, but as is often the case with graphical methods, the accuracy is poor. Kuske, indeed, recommended that the method be combined with an analytical calculation. For this purpose, Kuske gave a set of differential equations [11], [28], which do not however, appear to be as suitable as the other analytical methods at our disposal.

2.2.2. Analytical presentations

Mueller calculations

Section 1.1 shows how the state of polarization can be described at a point on the sphere, when the coordinates of the point are three of the components of Stokes' vector. Later (section 2.2.1), it was shown how the transformation from one layer into another can be described by means of a geometrical construction on the sphere.

Although Stokes' vector was introduced in 1852, H. Mueller [27], in 1946, was the first to introduce an operator - Mueller's matrix - corresponding to the rotation of the sphere, whereby the transition could be determined analytically.

Applying the following notation:

$\bar{V}_o$ and $\bar{V}_1$ are Stokes' vector before and after the passage layer No. 1,

$\bar{\bar{M}}_1$ is Mueller's matrix for layer No. 1,

we find that Mueller's calculations take the following form:

$$\bar{V}_1 = \bar{\bar{M}}_1 \bar{V}_o \tag{2.7}$$

In the case of the passage of several layers, we get after the n'th layer:

$$\bar{V}_n = \bar{\bar{M}}_n \bar{\bar{M}}_{n-1} \cdots\cdots \bar{\bar{M}}_1 \bar{V}_o = \bar{\bar{M}}_n' \bar{V}_o \tag{2.8}$$

The matrices $\bar{\bar{M}}_1 \cdots\cdot \bar{\bar{M}}_n$ correspond to linear retarders, assuming, as before, constant principal stress directions within the individual layers. The matrix $\bar{\bar{M}}_n'$, which gives the total effect of the n layers, will, in the general case, correspond to an elliptical retarder.

Mueller's matrix has the following form [14]:

$$\bar{\bar{M}}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & D^2-E^2+G^2 & 2DE & -2EG \\ 0 & 2DE & -D^2+E^2+G^2 & 2DG \\ 0 & 2EG & -2DG & 2G^2-1 \end{bmatrix} \tag{2.9a}$$

where

$$D = \cos 2\varphi_1 \sin \frac{\Delta_1}{2}$$

$$E = \sin 2\varphi_1 \sin \frac{\Delta_1}{2} \qquad (2.9b)$$

$$G = \cos \frac{\Delta_1}{2}$$

φ_1 is the orientation of the layer,

Δ_1 is the retardation of the layer.

In the form shown here, Mueller's matrix thus contains only information on two independent quantities, φ and Δ.

From (1.10), (2.7) and (2.9a-b) we get:

$$\bar{V}_1 = \bar{\bar{M}}_1 \bar{V}_o =$$

$$\begin{bmatrix} 1 \\ \cos 2\omega_o[\cos 2\beta_o-(1-\cos\Delta_1)\sin 2\varphi_1 \sin 2(\varphi_1-\beta_o)]-\sin 2\omega_o \sin\Delta_1 \sin 2\varphi_1 \\ \cos 2\omega_o[\sin 2\beta_o+(1-\cos\Delta_1)\cos 2\varphi_1 \sin 2(\varphi_1-\beta_o)]+\sin 2\omega_o \sin\Delta_1 \cos 2\varphi_1 \\ \cos 2\omega_o \sin\Delta_1 \sin 2(\varphi_1-\beta_o)+\sin 2\omega_o \cos\Delta_1 \end{bmatrix} \qquad (2.10)$$

A comparison with (2.5) will corroborate (2.10).

We will now apply Mueller calculations to demonstrate the existence of the characteristic directions.

As the matrix for each layer has the form,

$$\bar{\bar{M}}_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m_{22} & m_{33} & m_{24} \\ 0 & m_{32} & m_{33} & m_{34} \\ 0 & m_{42} & m_{43} & m_{44} \end{bmatrix}$$

$\bar{\bar{M}}'_n$ also has the form

$$\bar{\bar{M}}'_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m'_{22} & m'_{23} & m'_{24} \\ 0 & m'_{32} & m'_{33} & m'_{34} \\ 0 & m'_{42} & m'_{43} & m'_{44} \end{bmatrix}$$

If the incident light-wave $\bar{V}_o$ is linearly polarized with the orientation β_o, the emerging light-wave will be:

$$\bar{V}_n = \bar{\bar{M}}'_n \bar{V}_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & m'_{22} & m'_{23} & m'_{24} \\ 0 & m'_{32} & m'_{33} & m'_{34} \\ 0 & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 1 \\ \cos 2\beta_o \\ \sin 2\beta_o \\ 0 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ m'_{22} \cos 2\beta_o + m'_{23} \sin 2\beta_o \\ m'_{32} \cos 2\beta_o + m'_{33} \sin 2\beta_o \\ m'_{42} \cos 2\beta_o + m'_{43} \sin 2\beta_o \end{bmatrix} = \begin{bmatrix} 1 \\ \cos 2\omega_n \cos 2\beta_n \\ \cos 2\omega_n \sin 2\beta_n \\ \sin 2\omega_n \end{bmatrix} \quad (2.11)$$

$\bar{V}_n$ is linearly polarized if $\sin 2\omega_n = 0$, i.e.

$$m'_{42}\cos 2\beta_o + m'_{43}\sin 2\beta_o = 0, \text{ i.e.}$$
$$tg2\beta_o = -\frac{m'_{42}}{m'_{43}} \quad (2.12)$$

(2.12) has two solutions β_{o1} and β_{o2}, which correspond to two directions at right-angles to each other. From (2.11) we get the orientation β_n of the emerging, linearly polarized wave:

$$\cos 2\beta_n = m'_{22}\cos 2\beta_o + m'_{23}\sin 2\beta_o$$
$$\sin 2\beta_n = m'_{32}\cos 2\beta_o + m'_{33}\sin 2\beta_o \quad (2.13)$$

(2.13) has two solutions β_{n1} for $\beta_o = \beta_{o1}$ and β_{n2} for $\beta_o = \beta_{o2}$, which also correspond to two directions at right-angles to each other.

For β_{o1}, β_{o2}, β_{n1} and β_{n2} Aben [13] has introduced the designations primary characteristic directions (β_{oi}) and secondary characteristic directions (β_{ni}).

As the m'_{ij} coefficients depend on the retardations Δ_1, $\Delta_2 \ldots \Delta_n$, which in turn depend on the wavelength of the light, the characteristic directions are a function of both the stress field and the wavelength of the light used.

For a linear retarder, the characteristic directions correspond to the isoclines, which are independent of the wavelength.

By means of Mueller calculations we can derive some photo-elastic properties of significance to scattered-light technique on 3-dimensional models. The matrix $\bar{\bar{M}}'_n$ corresponds to an elliptical retarder and thus contains the three independent quantities:

φ'_n the orientation

Δ'_n the retardation

ω'_n the ellipticity

The justification for these designations lies in the fact that $\sin 2\omega'_n = 0$ for a linear retarder and that φ'_n and Δ'_n then correspond to orientation and retardation for this retarder.

It can be proved, [27], [14], that the matrix has the form

$$\bar{\bar{M}}'_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & D^2-E^2-F^2+G^2 & 2(DE+FG) & -2(DF+EG) \\ 0 & 2(DE-FG) & -D^2+E^2-F^2+G^2 & 2(DG-EF) \\ 0 & -2(DF-EG) & -2(DG+EF) & -D^2-E^2+F^2+G^2 \end{bmatrix}$$

where

$$D = \cos 2\omega'_n \cos 2\varphi'_n \sin \frac{\Delta'_n}{2}$$

$$E = \cos 2\omega'_n \sin 2\varphi'_n \sin \frac{\Delta'_n}{2}$$

$$F = \sin 2\omega'_n \sin \frac{\Delta'_n}{2}$$

$$G = \cos \frac{\Delta'_n}{2}$$

If we know $\bar{V}_n$ and can determine $\bar{V}_{n+1}$ then from

$$\bar{V}_{n+1} = \bar{\bar{M}}_{n+1}\bar{V}_n$$

compared with expression of the form (2.10), we can determine φ_{n+1} and Δ_{n+1}, i.e. determine the secondary principal direction and stress difference in layer No. n+1. By simple calculation, considering the n+1'th layer instead of the first layer, we get from (2.10):

$$\mathrm{tg}2\varphi_{n+1} = -\frac{\cos 2\omega_n \cos 2\beta_n - \cos 2\omega_{n+1} \cos 2\beta_{n+1}}{\cos 2\omega_n \sin 2\beta_b - \cos 2\omega_{n+1} \sin 2\beta_{n+1}} \qquad (2.14)$$

(2.14) can also be derived geometrically from the j-circle. The retardation Δ_{n+1} can then, as shown in [12], be determined from

$$\sin 2\omega_{n+1} = \cos 2\omega_n \sin 2(\varphi_{n+1}-\beta_n)\sin\Delta_{n+1} + \sin 2\omega_n \cos\Delta_{n+1}$$

Jones' calculations

Jones' calculations are carried out with matrices, as Mueller calculations; but Stokes' vector is replaced by Jones' vector (1.11), and Mueller's matrix by Jones' matrix:

$$\bar{\bar{J}}_1 = \bar{\bar{R}}(\varphi_1)\,\bar{\bar{N}}_1\bar{\bar{R}}(-\varphi_1) = \begin{bmatrix}\cos\varphi_1 & -\sin\varphi_1\\ \sin\varphi_1 & \cos\varphi_1\end{bmatrix}\begin{bmatrix}e^{-i\Delta_{11}} & 0\\ 0 & e^{-i\Delta_{12}}\end{bmatrix}\begin{bmatrix}\cos\varphi_1 & \sin\varphi_1\\ -\sin\varphi_1 & \cos\varphi_1\end{bmatrix}$$

$$\begin{bmatrix}e^{-i\Delta_{11}}\cos^2\varphi_1+e^{-i\Delta_{12}}\sin^2\varphi_1 & (e^{-i\Delta_{11}}-e^{-i\Delta_{12}})\cos\varphi_1\sin\varphi_1\\ (e^{-i\Delta_{11}}-e^{-i\Delta_{12}})\cos\varphi_1\sin\varphi_1 & e^{-i\Delta_{11}}\sin^2\varphi_1+e^{-i\Delta_{12}}\cos^2\varphi_1\end{bmatrix} \quad (2.15a)$$

Here, $\bar{\bar{R}}(-\varphi_1)$ gives a rotation of the coordinate system, so the light vector is described in a coordinate system with axes in the principal directions, $\bar{\bar{N}}_1$ determines the retardation of the components of the light vector in the principal directions, and $\bar{\bar{R}}(\varphi_1)$ rotates the coordinate system back to the fixed x-y-coordinate system. Δ_{11} and Δ_{12} are the retardation in the first and the second principal directions of layer No. 1, respectively. In a coordinate system with axes u and v in the secondary principal directions, Jones' matrix takes the form:

$$\bar{\bar{J}}_1 = \bar{\bar{N}}_1\bar{\bar{R}}(-\varphi) \quad (2.15b)$$

In the passage of several layers, we get, just as in Mueller calculations,

$$\bar{D}^*_n = \bar{\bar{J}}_n\ldots\bar{\bar{J}}_1\bar{D}^*_1 = \bar{\bar{J}}'_n\bar{D}^*_1 \quad (2.16)$$

where $\bar{\bar{J}}'_n$ gives the characteristics for the n layers.

If we let $\Delta_{11} = -\Delta_{12}$, corresponding to an isotropic phase displacement that does not affect interference phenomena between the two waves, we can, by inductive proof, conclude that $\bar{\bar{M}}'_n$ can be written

$$\bar{\bar{J}}'_n = \begin{bmatrix}e^{i\xi}\cos\theta & e^{i\zeta}\sin\theta\\ -e^{i\zeta}\sin\theta & e^{-i\xi}\cos\theta\end{bmatrix} \quad (2.17)$$

It can further be shown, [4]. [13], that the matrix $\bar{\bar{M}}'_n$ can also be written:

$$\bar{\bar{J}}_n = \begin{bmatrix} \cos\beta_{ni} & -\sin\beta_{ni} \\ \sin\beta_{ni} & \cos\beta_{ni} \end{bmatrix} \begin{bmatrix} e^{j\gamma_n} & 0 \\ 0 & e^{-j\gamma_n} \end{bmatrix} \begin{bmatrix} \cos\beta_{oi} & \sin\beta_{oi} \\ -\sin\beta_{oi} & \cos\beta_{oi} \end{bmatrix} \tag{2.18a}$$

$$= \bar{\bar{R}}(\beta_{ni})\bar{\bar{N}}'_n\bar{\bar{R}}(-\beta_{oi})$$

where

$$\begin{aligned} tg2\beta_{oi} &= \frac{\sin(\zeta+\xi)\sin2\theta}{\sin2\xi\cos^2\theta-\sin2\zeta\sin^2\theta} \\ tg2\beta_{ni} &= \frac{\sin(\zeta+\xi)\sin2\theta}{\sin2\xi\cos^2\theta+\sin2\zeta\sin^2\theta} \\ \cos2\gamma_n &= \tfrac{1}{2}[\cos2\xi+\cos2\zeta+\cos2\theta(\cos2\xi-\cos2\zeta)] \end{aligned} \tag{2.18b}$$

β_{oi} and β_{ni} are the characteristic directions discussed above, $i = 1,2$.

Jones' calculations have not been used very much in photo-elastic methods. Aben [13] has used the method to determine the state of stress in shells.

3. Dynamic descriptions

As stated in the introduction, the dynamic descriptions are based on the assumption of a relationship between the strains or stresses and the permitivity tensor ϵ_{ij}.

The subject matter of the next two pages forms the basis for the electromagnetic field theory for dielectrics and is described in textbooks on the theory of electricity and optics.

However, the concepts polarization of the matter, susceptibility tensor, dialectric tensor and dialectric constant are described in brief on pages A8-A9.

When the permitivity tensor is known, we can, in principle, solve Maxwell's electromagnetic field equations, i.e. determine the state of polarization as a function of time and place. The fact that the permitivity tensor is dependent on the strains is due to the anisotropic binding of the electrons of the matter during mechanical loading. This is demonstrated by means of the model in fig. 3.1, where the directions of the principal stresses coincide with those of the springs.

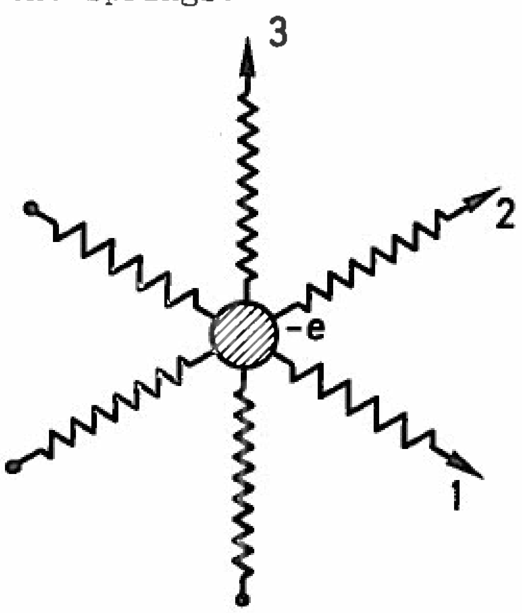


Fig. 3.1. Model of the anisotropic binding of the electrons during loading.

The polarization of the matter in a coordinate system with axes in the principal directions is as follows:

$$\bar{P} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} = \epsilon_o \begin{bmatrix} \chi_1 & 0 & 0 \\ 0 & \chi_2 & 0 \\ 0 & 0 & \chi_3 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \epsilon_o \bar{\bar{\chi}} \bar{E} \qquad (3.1a)$$

where $\bar{\bar{\chi}}$ is the susceptibility tensor, the elements of which are assumed to be linearly related to the stress of strain tensor. In an arbitrary coordinate system, (3.1a) has the form,

$$\bar{P} = P_i = \epsilon_o \begin{bmatrix} \chi_{11} & \chi_{12} & \chi_{13} \\ \chi_{21} & \chi_{22} & \chi_{23} \\ \chi_{31} & \chi_{32} & \chi_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \epsilon_o \chi_{ij} E_j \qquad (3.1b)$$

χ_{ij} is related to ϵ_{ij} by

$$\epsilon_{ij}\epsilon_o E_j = D_i = \epsilon_o E_i + P_i = (\delta_{ij} + \chi_{ij})\epsilon_o E_j \qquad (3.2)$$

from which

$$\epsilon_{ij} = \delta_{ij} + \chi_{ij}$$

where

$$\delta_{ij} = \begin{cases} 1 \text{ for } i = j \\ 0 \text{ for } i \neq j \end{cases}$$

Maxwell's equations, which form the basis for the following, have the form

$$\begin{aligned} &\text{rot } H_i = J_i + \frac{\partial D_i}{\partial t} \qquad &&\text{rot } E_i = -\frac{\partial B_i}{\partial t} \\ &\text{div } D_i = \rho &&\text{div } B_i = 0 \\ &D_i = \epsilon_{ij}\epsilon_o E_j \qquad B_i = \mu_{ij}\mu_o H_j &&I_i = \gamma_{ij} E_j \end{aligned} \qquad (3.3)$$

For a dielictric, the equations can be simplified, since

$$\mu_{ij} = \delta_{ij}$$
$$\gamma_{ij} = 0$$

as the material is non-magnetic and non-conductive. Maxwell's equations can thereby be reduced to

$$\begin{aligned} \epsilon_{ij}\epsilon_o \dot{E}_j &= \text{rot } H_i \\ -\mu_o \dot{H}_i &= \text{rot } E_i \\ \text{div } H_i &= 0 \end{aligned} \qquad (3.4)$$

The factors dealt with so far in this section are well-known from the electromagnetic field theory.

The first to use Maxwell's equations in photo-elasticity were Mindlin and Goodmann [17], who, in 1949, assumed the following relationship between ϵ_{ij} and the stress tensor σ_{ij}.

$$\epsilon_{ij}{}^{-1} = \frac{1}{n_o^2}\,\delta_{ij} + C_1\sigma_{ij} + C_2\sigma_{kk}\delta_{ij} \qquad (3.5)$$

where

n_o is the refractive index for the unloaded body, and

C_1 and C_2 are photo-elastic constants.

Later, O'Rourke [20], Aben [13] and Van Geen [19] formulated systems of equations corresponding to (3.4) and (3.5), from which they determined $\bar{E}$ and $\bar{H}$. Common to these dynamic solutions of the photo-elastic problem is the fact that the approximations necessary for the solution all lead back to the solution for the kinematical methods. Therefore, the only advantage of the dynamic methods is that we are made aware of the approximations concealed in the kinematic descriptions. Besides these, we will discuss two other approximations in our assumptions.

For high electric field strengths, the linear relationship between P_i and E_j (3.1) is not valid. However, we will not delve deeper into the question of the non-linear optical phenomena [9], but just note that, even with a laser, where 5-15 mW are distributed over some few mm^2, (3.1) provides a completely acceptable basis on which to work. (3.5) is an acceptable expression for the behaviour of most photo-elastic materials provided the materials are linear-elastic and there is no creep.

When the stresses are so great that the material shows plastic behaviour, or if the material creeps, (3.5) can no longer be used. However, the treatment of such problems lies outside the scope of this report. In order to avoid creep of yielding, materials with low creep must be selected, and the loads kept suitably small.

3.1 Description of Mindlin and Goodman

As early as 1940, Drucker and Mindlin [15], [16] formulated a photo-elastic theory based on the assumption that light moves in an elastic medium, the ether. The ether theory has little to do with reality, but it does lead to the same equations as the real dynamic theories, which are based on Maxwell's equations.

Drucker and Mindlin used the ether theory partly to determine an approximate expression for the state of polarization in the case in which the directions of the principal stresses only rotate a little [15], and partly to determine the stress field in plates [16].

We will now go through the main points of Mindlin's and Goodman's description [17].

With $(\epsilon_{ij})^{-1} = \pi_{ij}$, we get from (3.4) and (3.5):

$$\begin{aligned} \epsilon_o \mu_o \ddot{H}_i &= -\mathrm{rot}(\pi_{ij} \mathrm{rot}\, H_j) \\ \mathrm{div}\, H_i &= 0 \end{aligned} \qquad (3.6)$$

(3.6) are the basic equations governing the electromagnetic field of a light-wave in an optically heterogeneous and anisotropic body. This equation is written out in coordinates. It is shown that most of the terms are of minor magnitude, whereby the expression is considerably reduced.

Certain of the discarded terms correspond to the assumption that the light-waves propagates along straight lines despite the varying refractive indices, while others do not have such a simple explanation.

If we only consider a light-wave in the direction of the z-axis, and turn the coordinate system so that the axes coincide with the secondary principal directions, (3.6) reduces to

$$\begin{aligned} \frac{\partial u}{\partial z} + v\,\frac{\partial \varphi}{\partial z} &= \mp \frac{1}{\varkappa_2}\,\frac{\partial u}{\partial t} \\ \frac{\partial v}{\partial z} - u\,\frac{\partial \varphi}{\partial z} &= \mp \frac{1}{\varkappa_1}\,\frac{\partial v}{\partial t} \end{aligned} \qquad (3.7)$$

where

u and v are the magnitudes of the electrical field in the first and second principal direction, respectively.

$$\varkappa_1 = \frac{1}{n_1} \cdot \frac{1}{\sqrt{\epsilon_o \mu_o}}$$

$$\varkappa_2 = \frac{1}{n_2} \cdot \frac{1}{\sqrt{\epsilon_o \mu_o}}$$

n_1 is the refractive index for a light-wave with electric vector parallel to the first secondary principal direction.

φ is the angle from the x-axis to the first secondary principal direction.

If we insert in (3.7) solutions of the form

$$u = Ue^{j(\Omega t - \delta_1)}$$
$$v = Ve^{j(\Omega t - \delta_2)} \qquad (3.8)$$

where U = U(z) and V = V(z), we get

$$\frac{\partial U}{\partial z} - jU\frac{\partial \delta_1}{\partial z} - e^{j(\delta_1 - \delta_2)}V\frac{\partial \varphi}{\partial z} = -j\frac{\Omega}{k_2}U$$
$$\frac{\partial V}{\partial z} - jV\frac{\partial \delta_2}{\partial z} + e^{-j(\delta_1 - \delta_2)}U\frac{\partial \varphi}{\partial z} = -j\frac{\Omega}{k_1}V \qquad (3.9)$$

(3.9) are the basic photo-elastic equations in the form specified by Mindlin and Goodman.

The transition from (3.9) to Neumann's equations is comparatively simple and requires only basic mathematics, see [17]. Similarly, Aben's equations can easily be derived from (3.9), see [17] and [13].

Later, O'Rourke [20] and Kuske [21], also on the basis of dynamic assumptions, derived the photo-elastic basic equations, although in a form that seems less serviceable than (3.9). We will not go deeply into these descriptions, but turn to Aben [13], who, in 1966, gave a new formulation of the photo-elastic equations.

3.2 Aben's description 1966

Aben's assumptions are:

1) A system of differential equations derived from Ginsburg [13]:

$$\frac{\partial^2 E_1'}{\partial z^2} + \frac{\Omega^2}{\epsilon_o C^2} D_1' = 0$$
$$\frac{\partial^2 E_2'}{\partial z^2} + \frac{\Omega^2}{\epsilon_o C^2} D_2' = 0 \qquad (3.10)$$

where E_1', E_2', D_1' and D_2' are determined by

$$E_1 = E_1' e^{j\Omega t} \;,\quad E_2 = E_2' e^{j\Omega t}$$
$$D_1 = D_1' e^{j\Omega t} \;,\quad D_2 = D_2' e^{j\Omega t} \qquad (3.11)$$

(3.10) is obtained from Maxwell's equations, with certain approximations

2) $$D_i = \epsilon_{ij}\epsilon_o E_j \qquad (3.3)$$

3) $$\epsilon_{ij} = n_o^2\delta_{ij} + C_1\sigma_{ij} + C_2\sigma_{kk}\delta_{ij} \qquad (3.12)$$

(3.12) is the photo-elastic law in a form that was first given by O'Rourke [20] and that is more frequently used than (3.5).

We will not go through Aben's derivation here, since it is described in detail in [13], but will show how his result can be derived more easily from Jones' calculations:

In a coordinate system the axes of which are in the principal directions , Jones' calculations have the form

$$\bar{D}_i = \bar{D}_{i-1}+\Delta\bar{D}_{i-1} = \bar{\bar{N}}_i\bar{\bar{R}}(-(\varphi_i-\varphi_{i-1}))\bar{D}_{i-1} = \bar{\bar{N}}_i\bar{\bar{R}}(-\Delta\varphi_{i-1})\bar{D}_{i-1}$$
$$= \begin{bmatrix} e^{-j\Delta j_1} & 0 \\ 0 & e^{-j\Delta_{j2}} \end{bmatrix} \begin{bmatrix} \cos\Delta\varphi_{i-1} & \sin\Delta\varphi_{i-1} \\ -\sin\Delta\varphi_{i-1} & \cos\Delta\varphi_{i-1} \end{bmatrix} \bar{D}_{i-1}$$

We now neglect the time dependence $e^{j\Omega t}$ in Jones' vector by introducing E_1' and E_2':

$$\bar{D}_{i-1} = e^{j\Omega t}\begin{bmatrix} E_1' \\ E_2' \end{bmatrix} \qquad (3.13a)$$

and

$$\Delta D_{i-1} = e^{j\Omega t}\begin{bmatrix} \Delta E_1' \\ \Delta E_2' \end{bmatrix} \qquad (3.13b)$$

The photo-elastic law is:

$$\Delta_{i1} - \Delta_{i2} = C(\sigma_1-\sigma_2)$$

from which

$$\Delta_{i1} = \tfrac{1}{2}C(\sigma_1-\sigma_2) + \text{constant}$$
$$\Delta_{i2} = -\tfrac{1}{2}C(\sigma_1-\sigma_2) + \text{constant}$$

We can neglect an isotropic phase displacement, and therefore put the constant at 0.

If we further make use of the fact that

$$e^{j\Delta} = \cos\Delta + j\sin\Delta \approx 1 + j\Delta \text{ for } \Delta \to 0$$

and let

$$\Delta\varphi_i = d\varphi \to 0$$

we get:

$$\begin{bmatrix} E_1' + dE_1' \\ E_2' + dE_2' \end{bmatrix} = \begin{bmatrix} e^{-\frac{1}{2}jC(\sigma_1-\sigma_2)dz} & 0 \\ 0 & e^{\frac{1}{2}jC(\sigma_1-\sigma_2)dz} \end{bmatrix} \begin{bmatrix} 1 & d\varphi \\ -d\varphi & 1 \end{bmatrix} \begin{bmatrix} E_1' \\ E_2' \end{bmatrix} =$$

$$\begin{bmatrix} e^{-\frac{1}{2}jC(\sigma_1-\sigma_2)dz} & d\varphi e^{-\frac{1}{2}jC(\sigma_1-\sigma_2)dz} \\ -d\varphi e^{\frac{1}{2}jC(\sigma_1-\sigma_2)dz} & e^{\frac{1}{2}jC(\sigma_1-\sigma_2)dz} \end{bmatrix} \begin{bmatrix} E_1' \\ E_2' \end{bmatrix} =$$

$$\begin{bmatrix} 1 - \frac{1}{2}jC(\sigma_1-\sigma_2)dz & \\ -d\varphi & 1 + \frac{1}{2}jC(\sigma_1-\sigma_2)dz \end{bmatrix} \begin{bmatrix} E_1' \\ E_2' \end{bmatrix} \qquad (3.14)$$

For an arbitrary layer, we get from (3.14):

$$\begin{aligned} dE_1' &= -\tfrac{1}{2}jC(\sigma_1-\sigma_2)E_1'dz+E_2'd\varphi \\ dE_2' &= \tfrac{1}{2}jC(\sigma_1-\sigma_2)E_2'dz-E_1'd\varphi \end{aligned} \qquad (3.15)$$

The photo-elastic equations were first given in this form by Aben [13], who used it as the basis for his photo-elastic investigations.

The results derived by Aben from (3.15), the characteristic directions and the appearance of Jones' matrix for an arbitrary retarder, are described in this report under Jones and Mueller calculations.

Aben used (3.15) to determine stresses in shells and in a body in which there is constant rotation of the secondary principal stresses.

4. Assessment of the various descriptions

This report is divided into two main sections - kinematical and dynamic descriptions, depending on the assumptions on which the descriptions are based.

A study of the dynamic descriptions shows that certain approximations are necessary in order to arrive at a serviceable solution, but that these approximations are of much less importance than those otherwise connected with photo-elastic investigations.

It is therefore more reasonable to classify the methods according to function and possibilities of use. This is attempted in the following table, where the mutual relationship of the descriptions is also elucidated.

It is impossible to make a general statement as to which descriptions are the most serviceable. Poincaré's sphere provides an exellent means of forming a qualitative impression of the state of polarization or the state of stress (the difference between the secondary principal stresses and their orientation) by means of the scattered light method. If, on the other hand, we want a quantitative stress determination, the analytical description of Poincaré's sphere [34] or Mueller calculations [14] seem to provide the best tool, since these give a relationship between the difference between, and direction of, the secondary principal stresses and the light-ellipse, the shape and orientation of which can be measured direct.

However, the other analytical methods have also been used in the scattered light method (Jessop [33], Cheng [35] and Srinath [31]), but these methods are less general than those utilizing Poincaré's sphere.

State of polarization described by: 1) electric field 2) light-ellipse	Analytical methods Differential equations	Analytical methods Layers of finite thickness	Geometrical methods Layers of finite thickness
1a) electric field (u,v) in the secondary principal directions	Neumann (1841) Mindlin and Goodmann (1949) Aben (1964)	Jones calculations (1941)	
1b) electric field (x,y) in a coordinate system with fixed axes. (x,y) = the real part of Jones's vector	Aben (1966)	Jones calculations (1941)	
2a) Form and orientation of the light-ellipse in a coordinate system with fixed axes		Analytical description of Poincaré's sphere	Poincaré's sphere (1892) J-circle ~ plane projection of the sphere (Menges 1940) Stereographic projection of the sphere (Wulff's grid)
2b) Stokes' vector $\bar{V} = \begin{matrix} 1 \\ \cos 2\omega \cos 2\beta \\ \cos 2\omega \sin 2\beta \\ \sin 2\omega \end{matrix}$		Mueller calculations (1946)	

Conclusion

All the methods discussed permit determination of the state of polarization along a line when the secondary state of stress in a section perpendicular to the line is known throughout this.

However, it is the inverse problem that is of interest.

We know that the polarization field of the emerging light-wave can be characterized by means of the state of polarization of the incident light-wave and a function of the state of stress along the light-wave. This function (Jones matrix, Mueller matrix or the transformation on Poincaré's sphere) is characterized by three quantities. It is therefore impossible to determine the stress field along a line in the general case on the basis of known states of polarization of the incident and emerging light-waves, since all we determine here is the three quantities mentioned. Up to the present time, two alternative methods have been used:

1) Stress determination in bodies in which the stress field along a line is characterized by maximum three quantities besides the hydrostatic stress field. This procedure has been used particularly on discs and, since 1934, also on plates, shells and rotation-symmetrical bodies under special loads [13], [20], [22], [25], [37], et al.

   As an example, it can be mentioned that the disc and plate problem has two unknown quantities besides the sum of the principal stresses. This problem can be solved, although in the case of the plate problem, the plate must be analysed only on one side of the middle plane.

   In shell problem, we have four unknown quantities besides the sum of the principal stresses. This problem can, for example, be solved in the same way as the plate problem, when both sides of the middle plane are analysed [9]. However, there are several solutions to the shell problem [13].

2) Determination of the difference between the secondary principal stresses and of the secondary principal directions by mechanical slicing (Oppel 1936 [38]) or optical slicing (Weller 1939 [39] and Menges 1940 [8]) in such thin slices that an assumption can be made of constant principal stress

direction within the slice. By using this technique in three planes at the same point, we can determine the deviation from the hydrostatic stress field, i.e. five of the sex parameters of the stress field can be ascertained.

The remaining parameter can, for example, be determined by integrating the equilibrium equations from a known boundary condition.

With the development of the slicing technique and the appearance of sensitive light-detectors, these methods have been greatly refined and improved in recent years, a process which, particularly as regards the scattered light method, is not yet ended.

Besides the methods mentioned, there are two entirely new principles offering interesting possibilities for a three-dimensional stress determination by means of known state of polarization before and after the passage of the model.

3) When the model is subjected to a magnetic or electrical field, Maxwell's electromagnetic field equations are altered, and in this way further information can be obtained on the stress field. Aben has described the application of this principle to plates [36].

4) By letting the light-wave pass in various directions, we can similarly obtain further information on the stress field. However, no attempt seems to have been made to clarify the mathematical problems arising in the general, three-dimensional case, although Drucker and Woodward [26] have formulated this problem and shown that it's solution requires $5n^3$ different directions, imagining the body to be divided in n^3 small boxes within which the stress field is assumed to be constant. See, further [40], in which examples are given of the use of holography for the registration of the photo-elastic effect of light-waves in several directions.

NOTATION

Symbol	Explanation	Fig. or equ.
a	$a_1^2 + a_2^2$	Fig. 1.1
a_1, a_2	The Vectors of the electrical amplitudes in a u-v or x-y coordinate system	Fig. 1.1
a_1', a_2'	a_1 og a_2 after passage of a birefringent layer	
B_i	Magnetic induction, i = 1,3	3.3
C, C_1, C_2	Photo-elastic constants	3.5, 3.14
D, E, F, G	Quantities included in Mueller matrix	2.9a
$\bar{D}$	Jones vector in a u-v coordinate system	1.11a
$\bar{D}^*$	Jones vector after passage of layer No. n	2.16
D_1', D_2'	Complex amplitude of electrical displacement	3.11
D_i	Electrical induction, i=1,3	3.2
d_1, d_2	Principal axes of the light-ellipse	Fig. 1.1
$\bar{E}$, E_i	Electrical field, i=1.3	3.1
E_1', E_2'	Complex amplitude of electrical field	3.13
H	Point on Poincaré's sphere corresponding to linearly polarized light in the direction of the x-axis	Fig. 1.2, Fig. 2.2
$\bar{H}$, H_i	Magnetic field, i=1,3	3.3
J_i	Electrical current vector,i=1,3	3.3
j^i	Imaginary unit 1.11a	
$\bar{\bar{J}}_n$	Jones matrix	2.15
k_1, k_2	Strain-optical constants	2.1
$\bar{\bar{M}}_i$	Mueller or Jones matrix for layer No. i	2.7
$\bar{\bar{M}}_i'$	Mueller of Jones matrix for i layers	2.8

m_{ij}, m'_{ij}	Coefficients in Mueller matrix	2.11
$\bar{\bar{N}}_i$	Matrix giving the retardation of layer No. i	2.15
n_o	Refractive index for unloaded body	3.5
n_i	Refractive index, i=1,3	
O	Centre of Poincaré's sphere and of the j-circle	Fig. 2.2 Fig. 2.3
P, P_i	Point on Poincaré's sphere corresponding to type of polarization after passage of layer No. i	Fig. 1.2
$\bar{P}$, P_i	Polarization of the matter, i=1,3	3.1
P'_i	P projected on horizontal plane	Fig. 2.3
P''_i	P rotated downwards in horizontal plane	Fig. 2.3
$\bar{\bar{R}}(\varphi)$	Rotation matrix	2.15
S_i	Stokes' vector in an x,y-coordinate system, i=0,3	1.8, 1.10
t	Time unit	1.1
U, V	Amplitude of magnetic field in u,v-coordinate system	3.8
V	Point on Poincaré's sphere corresponding to linearly polarized light in direction of y-axis	Fig. 1.2
$\bar{V}$, $\bar{V}_n$	Stokes' vector after layer No. n	
u, v	Secondary principal axes	Fig. 1.1 Fig. 2.1
u, v	Electrical or magnetic field in u,v-coordinate system	
u', v'	u and v-axes after infinitesimal rotation dφ	Fig. 2.1
v_o	Velocity of light in unloaded condition	2.1
v_a, v_b, v_c	Velocities of light in principal directions	2.1
x, y, z	Fixed coordinate system, x corresponding to horizontal, z to direction of propagation of the light-wave	Fig. 1.1

x, y, z	Coordinates on Poincaré's sphere	1.7
x, y	Electrical or magnetic field in x,y-coordinate system	1.1
α	$tg\alpha = a_1/a_2$	1.4, Fig. 1.1
α'	$tg\alpha' = a_1'/a_2'$	
β, β_i	Angle from x-axis to principal axis of the ellipse after layer No. i	Fig. 1.1 Fig. 2.2
β'	Angle from u-axis to principal axis of ellipse	Fig. 1.1
β_{oi}, β_{ni}	Primary and secondary characteristic directions, i= 1,2	2.12, 2.13
γ_n	Half the retardation of an elliptical retarder of n layers	2.18
γ_{ij}	Conductivity tensor	3.3
Δ	Difference sign	
Δ, Δ_i	Retardation for layer No. i	2.4, Fig. 2.2
Δ_{i1}, Δ_{i2}	Retardation in first and second principal direction, respectively, for layer No. i	3.13
Δ_n'	$= 2 \cdot \gamma_n$ = retardation of an elliptical retarder	
δ	Retardation of light-wave in u,v-coordinate system	1.1
δ'	δ after passage of a birefringent layer	
δ_1, δ_2	Retardation of light-wave in first and second principal direction, respectively	2.2
δ_1', δ_2'	δ_1 and δ_2 after infinitesimal rotation $d\varphi$ of the secondary principal directions	11
δ_1', δ_2'	Retardations in a u,v-coordinate system after passage of a birefringent layer	
δ_{ij}	Kronecker's tensor	
δ^*	Retardation of light-wave in an x,y-coordinate system	1.1

ϵ_o	Permitivity of vacuum	3.1
ϵ_1, ϵ_2, ϵ_3	Principal strains	2.1
ϵ_{ij}	Permitivity tensor	3.2
ζ, θ, ξ	Quantities included in Jones vector for an elliptical retarder	2.17
$\varkappa_1$, $\varkappa_2$	Velocities of light with electric vibrations in principal directions	
μ_o	Magnetic permitivity of vacuum	3.3
μ_{ij}	Magnetic permitivity tensor	3.3
ξ	See under ζ	
σ_{ij}	Stress tensor	3.5
σ_1, σ_2	Secondary principal stresses	3.14
φ, φ_i	Angle from horizontal to first principal stress direction for layer No. i	Fig. 2.1 2.9b
φ_n'	Orientation of an elliptical retarder with n layers	
$\varkappa_1$, $\varkappa_2$, $\varkappa_3$	Elements of the susceptibility tensor in a coordinate system with axes in the principal directions	3.1a
$\bar{\bar{\varkappa}}$, $\varkappa_{ij}$	Susceptibility tensor	3.1b
ω	Ellipticity of light ellipse	Fig. 1.1
Ω	Angular frequency	1.1
ω_i	Ellipticity after layer No. i	
ω_n'	Ellipticity for a retarder of n layers	

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Appendix.

1.1 Determination of the orientation and the shape of the light-ellipse.

$$x = a_1 \cos\Omega t$$

$$y = a_2 \cos(\Omega t + \delta^*) = a_2(\cos\Omega t \cos\delta^* + \sin\Omega t \sin\delta^*)$$

$$\left(\frac{x}{a_1}\right)^2 = \cos^2\Omega t$$

$$\left(\frac{y}{a_2} - \cos\Omega t \cos\delta^*\right)^2 = \left(\frac{y}{a_2} - \frac{x}{a_1}\cos\delta^*\right)^2 =$$

$$\sin^2\Omega t \sin^2\delta = \left[1 - \left(\frac{x}{a_1}\right)^2\right]\sin^2\delta$$

from which

$$\left(\frac{x}{a_1}\right)^2 + \left(\frac{y}{a_2}\right)^2 - 2\,\frac{xy}{a_1 a_2}\cos\delta^* = \sin^2\delta^* \qquad (1)$$

The roots of the determinant are found

$$\begin{vmatrix} \frac{1}{a_1^2} - \lambda & -\frac{\cos\delta^*}{a_1 a_2} \\ -\frac{\cos\delta^*}{a_1 a_2} & \frac{1}{a_2^2} - \lambda \end{vmatrix} = \lambda^2 - \left(\frac{1}{a_1^2} + \frac{1}{a_2^2}\right)\lambda + \frac{\sin^2\delta^*}{a_1 a_2^1} = 0$$

$$\lambda = \frac{1}{2a_1^2 a_2^2}\left(a_1^2 + a_2^2 \pm \sqrt{(a_1^2+a_2^2)^2 - 4a_1^2 a_2^2 \sin^2\delta^*}\right. \qquad (2)$$

From (1) and (2) we get the semi axes of the ellipse

$$\left.\begin{matrix} d_1 \\ d_2 \end{matrix}\right\} = \sqrt{\frac{\sin^2\delta^*}{\lambda}} = \sqrt{\frac{2a_1^2 a_2^2 \sin^2\delta^*}{a_1^2 + a_2^2 \pm \sqrt{(a_1^2+a_2^2)^2 - 4a_1^2 a_2^2 \sin^2\delta^*}}} =$$

$$\sqrt{\frac{a_1^2 + a_2^2}{2}\left(1 \pm \sqrt{1 - \frac{4a_1^2 a_2^2}{(a_1^2+a_2^2)^2}\sin^2\delta^*}\right)} =$$

$$\sqrt{\frac{a^2}{\sqrt{2}}\left(1 \pm \sqrt{1 - \sin^2 2\alpha \sin^2\delta^*}\right)} \qquad (3)$$

where

$$a_1 = a\cos\alpha \quad \text{og} \quad a_2 = a\sin\alpha \tag{4}$$

The orientations of the axes of the light ellipse are determined. The direction vector to the semi axes forms an angle β with the x-axes, where

$$\mathrm{tg}\beta = \frac{\frac{1}{a_1^2} - \lambda_1}{\frac{\cos\delta^*}{a_1 a_2}} = \frac{\mathrm{tg}\alpha - \frac{1-\sqrt{1-\sin^2 2\alpha \sin^2\delta^*}}{\sin 2\alpha}}{\cos\delta^*} =$$

$$\frac{1-\cos 2\alpha - 1 + \sqrt{1-\sin^2 2\alpha\sin^2\delta^*}}{\cos\delta^*\sin 2\alpha} =$$

$$\frac{-\cos 2\alpha + \sqrt{1-\sin^2 2\alpha\sin^2\delta^*}}{\cos\delta^*\sin 2\alpha}$$

$$\mathrm{tg}2\beta = \frac{2\mathrm{tg}\beta}{1-\mathrm{tg}^2\beta} =$$

$$\frac{2\cos\delta^*\sin 2\alpha(-\cos 2\alpha + \sqrt{1-\sin^2 2\alpha\sin^2\delta^*})}{\cos^2\delta^*\sin^2 2\alpha - \cos^2 2\alpha - 1 + \sin^2 2\alpha\sin^2\delta^* + 2\cos 2\alpha\sqrt{1-\sin^2 2\alpha\sin^2\delta^*}}$$

$$= \frac{2\cos\delta^*\sin 2\alpha(-\cos 2\alpha + \sqrt{1-\sin^2 2\alpha\sin^2\delta^*})}{-2\cos^2\alpha + 2\cos 2\alpha\sqrt{1-\sin^2 2\alpha\sin^2\delta^*}} =$$

$$\cos\delta^*\mathrm{tg}2\alpha \tag{5}$$

1.2 <u>Demonstration that Stoke's vector corresponds to the coordinates on Poincaré's sphere.</u>

From (4) we get

$$a_1^2 + a_2^2 = a^2 \tag{6a}$$

$$a_1^2 - a_2^2 = a^2\cos 2\alpha \tag{6b}$$

$$2a_1 a_2 = a^2\sin 2\alpha \tag{6c}$$

From (3)

$$d_1^2 + d_2^2 = a^2$$

$$d_1 d_2 = \frac{a^2}{2} \sqrt{(1-1+\sin^2 2\alpha \sin^2 \delta^*)} = \frac{a^2}{2} \sin 2\alpha \sin\delta^*$$

From $tg\omega = \frac{d_2}{d_1}$ we get

$$d_1 d_2 = a^2 \cos\omega \sin\omega = \frac{a^2}{2} \sin 2\omega$$

i.e.

$$\sin 2\omega = \sin 2\alpha \sin\delta^* \qquad (7)$$

From (5) and (7) we get

$$1 = \sin^2\delta^* + \cos^2\delta^* = \frac{\sin^2 2\omega + \cos^2 2\alpha tg^2 2\beta}{\sin^2 2\alpha}$$

$$\sin^2 2\alpha = 1-\cos^2 2\alpha = \sin^2 2\omega + \cos^2 2\alpha tg^2 2\beta$$

from which

$$\cos 2\alpha = \pm \cos 2\omega \cos 2\beta \qquad (8)$$

From (6a):

$$S_o = a_1^2 + a_2^2 = a^2$$

From (6b) and (8):

$$S_1 = a_1^2 - a_2^2 = a^2 \cos 2\omega \cos 2\beta \qquad (9)$$

From (9), (6b), (6c) and (5)

$$S_2 = 2a_1 a_2 \cos\delta^* = S_1 \cdot \frac{2a_1 a_2}{a_1^2 - a_2^2} \cos\delta^* =$$

$$a^2 \cos 2\omega \cos 2\beta \cdot tg 2\alpha \cos\delta^* = a^2 \cos 2\omega \sin 2\beta$$

From (6c) and (7)

$$S_3 = 2a_1 a_2 \sin\delta^* = a^2 \sin 2\alpha \cdot \frac{\sin 2\omega}{\sin 2\alpha} = a^2 \sin 2\omega$$

2.1 Neumann's derivation of 2.3 - 2.4.

Before the passage we have, (the electric vector) in the (secondary) principal directions,

$$\left.\begin{aligned} u &= a_1\cos(\Omega t+\delta_1) \\ v &= a_2\cos(\Omega t+\delta_2) \end{aligned}\right\} \quad (10)$$

and after the passage,

$$\left.\begin{aligned} u' &= a_1' \cos(\Omega t+\delta_1^1) \\ v' &= a_2' \cos(\Omega t+\delta_2^1) \end{aligned}\right\} \quad (11)$$

First, let us consider the case where there is solely a rotation of the secondary principal axes, and where $\varepsilon_1 - \varepsilon_2 = 0$ in the layer. We then get, see fig. 3,

$$\left.\begin{aligned} u' &= a_1\cos d\varphi\cos(\Omega t+\delta_1) + a_2\sin d\varphi\cos(\Omega t+\delta_2) = \\ &(a_1\cos d\varphi+a_2\cos(\delta_2-\delta_1)\sin d\varphi)\cos(\Omega t+\delta_1) \\ &-a_2\sin(\delta_2-\delta_1)\sin d\varphi\sin(\Omega t+\delta_1) = \\ &a_1'\cos(\Omega t+\delta_1') \\ v' &= a_2\cos d\varphi\cos(\Omega t+\delta_2) - a_1\sin d\varphi\cos(\Omega t+\delta_1) = \\ &(a_2\cos d\varphi-a_1\cos(\delta_2-\delta_1)\sin d\varphi)\cos(\Omega t+\delta_2) \\ &-a_1\sin(\delta_2-\delta_1)\sin d\varphi\sin(\Omega t+\delta_2) = \\ &a_2'\cos(\Omega t+\delta_2') \end{aligned}\right\} \quad (12)$$

As $d\varphi \ll 1$, we get

$$(a_1')^2 = a_1^2+2a_1a_2\cos(\delta_2-\delta_1)d\varphi$$

$$(a_2')^2 = a_2^2-2a_1a_2\cos(\delta_2-\delta_1)d\varphi$$

from which we get as $a_1 \approx a_1'$ and $a_2 \approx a_2'$

$$\left.\begin{aligned} da_1 &= a_1'-a_1 = a_2\cos(\delta_2-\delta_1)d\varphi \\ da_2 &= a_2'-a_2 = -\, a_1\cos(\delta_2-\delta_1)d\varphi \end{aligned}\right\} \quad (13)$$

In order to determine δ_1' and δ_2' Neumann formulates the following equations, the meaning of which appears by writing u and v in complex form and then rotating the coordinate system

$$\left.\begin{aligned} a_1'\cos\delta_1' &= a_1\cos d\varphi\cos\delta_1 + a_2\sin d\varphi\cos\delta_2 \\ a_1'\sin\delta_1 &= a_1\cos d\varphi\sin\delta_1 + a_2\sin d\varphi\sin\delta_2 \\ a_2'\cos\delta_2' &= -a_1\sin d\varphi\cos\delta_1 + a_2\cos d\varphi\cos\delta_2 \\ a_2'\sin\delta_2' &= -a_1\sin d\varphi\sin\delta_1 + a_2\cos d\varphi\sin\delta_2 \end{aligned}\right\} \quad (14)$$

From (13) we get

$$a_1'a_2'\sin(\delta_2'-\delta_1') = a_1 a_2\sin(\delta_2-\delta_1)$$

from which

$$d(\delta_2-\delta_1) = -\mathrm{tg}(\delta_2-\delta_1)\,\frac{d(a_1 a_2)}{a_1 a_2}$$

which, together with (13), leads to

$$\frac{\delta(\delta_2-\delta_1)}{\delta\varphi} = -\sin(\delta_2-\delta_1)\,\frac{-a_1^2+a_2^2}{a_1 a_2} = +2\sin(\delta_2-\delta_1)\cot 2\alpha \quad (15)$$

in that we only consider the variation in φ, and as $\mathrm{tg}\alpha = \frac{a_1}{a_2}$.
By adding the contribution from the difference in principal strain we get the total increment of the phase retardation:

$$d\delta = C(\varepsilon_1-\varepsilon_2)dz+2\cot 2\alpha\sin\delta d\varphi \quad (16)$$

As an alternative to this procedure, (15) can be derived from (12) as done by Coker and Filon [7] and later by Jessop [33].

2.2.1 Geometrical descriptions.

Proof of the transformation on Poincaré's sphere. The proof falls in two sections:

1. Determination of the coordinates of P_1
2. Proof of the indentity between P_1 and the new state of polarization.

1: According to spherical geometry we have

$$\cos P_1OR = \cos P_oOR = \cos 2\omega_o \cos 2(\varphi_1 - \beta_o)$$

$$\sin P_1OR = \pm \sqrt{1-\cos^2 2\omega_o \cos^2 2(\varphi_1-\beta_o)}$$

$$\sin P_oRQ_o = \frac{\sin 2(\varphi_1-\beta_o)\cos 2\omega_o}{\pm\sqrt{1-\cos^2 2\omega_o\cos^2 2(\varphi_1-\beta_o)}}$$

$$\sin P_1RQ_1 = \sin(P_oRQ_o+\Delta_1) = \frac{\sin 2\omega_o\cos\Delta_1+\sin 2(\varphi_1-\beta_o)\cos 2\varphi_o\sin\Delta_1}{\pm\sqrt{1-\cos^2 2\omega_o\cos^2 2(\varphi_1-\beta_o)}}$$

$$\cos P_1RQ_1 = -\cos(P_oRQ_o+\Delta_1) = \frac{\sin 2\omega_o\sin\Delta_1-\sin 2(\varphi_1-\beta_o)\cos 2\omega_o\cos\Delta_1}{\pm\sqrt{1-\cos^2 2\omega_o\cos^2 2(\varphi_1-\beta_o)}}$$

$$\sin 2\omega_1 = \sin P_1RQ_1 \sin P_1OR = \sin 2\omega_o\cos\Delta_1+\sin 2(\varphi_1-\beta_o)\cos 2\omega_o\sin\Delta_1 \qquad (17)$$

$$\mathrm{tg}2(\beta_1-\varphi_1) = \cos P_1RQ_1\,\mathrm{tg}P_1OR = \frac{\sin 2\omega_o\sin\Delta_1-\sin 2(\varphi_1-\beta_o)\cos 2\omega_o\cos\Delta_1}{\cos 2\omega_o\cos 2(\varphi_1-\beta_o)} \qquad (18)$$

From (18) we get

$$\mathrm{tg}2\beta_1 = \frac{\mathrm{tg}2(\beta_1-\varphi_1)+\mathrm{tg}2\varphi_1}{1-\mathrm{tg}2(\beta_1-\varphi_1)\mathrm{tg}2\varphi_1}$$

$$\frac{\sin 2\omega_o\sin\Delta_1\cos 2\varphi_1-\sin 2(\varphi_1-\beta_o)\cos 2\omega_o\cos\Delta_1\cos 2\varphi_1+\cos 2\omega_o\cos 2(\varphi_1-\beta_o)\sin 2\varphi_1}{\cos 2\omega_o\cos 2(\varphi_1-\beta_o)\cos 2\varphi_1-\sin 2\varphi_1\sin\Delta_1\sin 2\varphi_1+\sin 2(\varphi_1-\beta_o)\cos 2\omega_o\cos\Delta_1\sin 2\varphi_1}$$

$$\frac{\cos 2\omega_o[\sin 2\beta_o+(1-\cos\Delta_1)\cos 2\varphi_1\sin 2(\varphi_1-\beta_o)]+\sin 2\ {}_o\sin\Delta_1\cos 2\varphi_1}{\cos 2\omega_o[\cos 2\beta_o-(1-\cos\Delta_1)\sin 2\varphi_1\sin 2(\varphi_1-\beta_o)]-\sin 2\omega_o\sin\Delta_1\cos 2\varphi_1} \qquad (19)$$

2: Before the passage af the layer we have two vibrations in the directions of the semi axes of the light-ellipse:

$$E_1 = d_1\cos\Omega t$$

$$E_2 = d_2\cos(\Omega t + \frac{\pi}{2}) = d_2\sin\Omega t$$

After the passage of the layer with retardation Δ_1 and orientation φ, we have the following two vibrations in the principal directions of the layer, in that $d_1 = \cos\omega_o$ and $d_2 = \sin\omega_o$:

$$u = a_1'\cos(\Omega t+\delta_1') =$$

$$\cos\omega_o\cos(\varphi_1-\beta_o)\cos\Omega t+\sin\omega_o\sin(\varphi_1-\beta_o)\sin\Omega t$$

$$v = a_2'\cos(\Omega t+\delta_2') =$$

$$\sin\omega_o\cos(\varphi_1-\beta_o)\sin(\Omega t+\Delta_1)-\cos\omega_o\sin(\varphi_1-\beta_o)\sin(\Omega t+\Delta_1)$$

$$= [\sin\omega_o\cos\Delta_1\cos(\varphi_1-\beta_o)+\cos\omega_o\sin\Delta_1\sin(\varphi_1-\beta_o)]\sin\Omega t$$

$$+ [\sin\omega_o\sin\Delta_1\cos(\varphi_1-\beta_o)-\cos\omega_o\cos\Delta_1\sin(\varphi_1-\beta_o)]\cos\Omega t$$

The corresponding retardations are determined by:

$$\sin\delta_1' = -\frac{1}{a_1'}\sin\omega_o\ \sin(\varphi_1-\beta_o)$$

$$\cos\delta_1' = \frac{1}{a_1'}\cos\omega_o\ \cos(\varphi_1-\beta_o)$$

$$\sin\delta_2' = -\frac{1}{a_2'}[\sin\omega_o\cos\Delta_1\cos(\varphi_1-\beta_o)+\cos\omega_o\sin\Delta_1\sin(\varphi_1-\beta_o)]$$

$$\cos\delta_2' = \frac{1}{a_2'}[\sin\omega_o\sin\Delta_1\cos(\varphi_1-\beta_o)-\cos\omega_o\cos\Delta_1\sin(\varphi_1-\beta_o)]$$

from which

$$\cos\delta' = \cos(\delta_2'-\delta_1') = \cos\delta_2'\cos\delta_1'+\sin\delta_2'\sin\delta_1' =$$

$$\frac{1}{a_1'a_2'}\,[\cos\omega_o\sin\omega_o\sin\Delta_1-\cos2\omega_o\cos\Delta_1\cos(\varphi_1-\beta_o)\sin(\varphi_1-\beta_o)] =$$

$$\frac{1}{2a_1'a_2'}\,[\sin2\omega_o\sin\Delta_1-\cos2\omega_o\cos\Delta_1\sin2(\varphi_1-\beta_o)]$$

The difference between the squares of the amplitudes becomes:

$$(a_1')^2-(a_2')^2 = \cos2\omega_o\cos2(\varphi_1-\beta_o)$$

Applying (1.5) to a coordinate system with axes in the principal directions of the layer we get:

$$\mathrm{tg}2(\beta_1-\varphi_1) = \cos\delta'\,\mathrm{tg}2\alpha'$$

$$\cos\delta' \cdot \frac{2a_1'a_2'}{(a_1')^2-(a_2')^2} =$$

$$\frac{\sin2\omega_o\sin\Delta_1-\cos2\omega_o\cos\Delta_1\sin2(\varphi_1-\beta_o)}{\cos2\omega_o\cos2(\varphi_1-\beta_o)}$$

which is in accordance with (18). The ellipticity ω_1 is determined from (1.3) as $\mathrm{tg}\omega_1 = d_2/d_1$:

$$\sin2\omega_1 = 2d_1d_2 =$$

$$\sin2\alpha'\sin\delta' =$$

$$2a_1'a_2' \cdot (\sin\delta_2'\cos\delta_1'-\cos\delta_2'\sin\delta_1') =$$

$$\sin2\omega_o\cos\Delta_1+\sin2(\varphi_1-\beta_o)\cos2\omega_o\sin\Delta_1$$

which is in accordance with (17).

3. The fundamental concepts of the electromagnetic theory for dielectrics.

It will be seen from fig. 3.1 that if the substance contains a resulting electric field $\bar{E}$, this will cause a displacement of the electrons relative to the nucleus. The displaced electrons form dipoles causing an electric field $\bar{E}_{pol}$. The resulting electric field $\bar{E}$ can be considered as the sum of two contributions: E_{pol} and a new field, the dielectric displacement $\bar{D}$, defined by

$$\bar{E} = \frac{1}{\epsilon_o} \bar{D} + \bar{E}_{pol}$$

where we have introduced a fundamental constant, the permitivity of vacuum:

$$\epsilon_o = 8,85 \cdot 10^{-12}$$

As the polarization $\bar{P}$ of the sustance is defined by

$$\bar{P} = - \epsilon_o \bar{E}_{pol}$$

we get 3.2:

$$\bar{D} = \epsilon_o \bar{E} + \bar{P}$$

The suceptibility tensor $\bar{\bar{X}}$, which is an expression of the substance's capacity for polarization is defined by 3.1. The dielectric tensor ϵ_{ij}, which also depends only on the substance, indicates the relationship between the $\bar{D}$-field and the $\bar{E}$-field. E_{ij} is defined by 3.2.

AFDELINGEN FOR BÆRENDE KONSTRUKTIONER
DANMARKS TEKNISKE HØJSKOLE

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