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ANALYSIS OF PRESTRESSED CABLE SYSTEMS SUPPORTED BY ELASTIC BOUNDARY STRUCTURES

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SUMMARY

Two different methods are presented for the analysis of cable systems supported by elastic boundary structures: 1. the displacement method applied to the complete structure (cable system plus boundary structure) and 2. a mixed method, in which the horizontal components of the cable forces are introduced as extra unknowns in addition to the generalized node displacements. The basic theory of the two methods is presented, and the governing nonlinear equations are derived. The Newton-Raphson iteration method is used for the solution in both cases.

Finally, a numerical example is presented (cable net supported by ring beam), and the results obtained by means of the above two methods of analysis are compared.

1. INTRODUCTION

The development of theoretical methods for the analysis of hanging roof structures began in about 196o. The early work on this subject was based on a treatment of the cable structure as an equivalent continuous system, for which governing differential or integro-differential equations were derived, see Schleyer [1o] and reference [6]. However, with the increasing use of computers it gradually became evident that more satisfactory results could be obtained by treating the cable structure as a discrete system. This method leads to a system of nonlinear equations with a finite number of unknowns, which can be solved efficiently by computer. Because of these advantages, the discrete approach has been used by the majority of research workers since about 1965. During the 196os, the theoretical work was devoted mainly to the analysis of plane cable structures and cable nets supported by rigid boundary structures, see for example references [2], [3], [7], [8], [11] and [12].

The more complex problem of the analysis of cable systems supported by elastic boundary structures is of considerable practical interest, as this type of structural system is often used in practical applications. Serious theoretical work on this type of problem began about four years ago, and it is only recently that satisfactory methods of solution have been developed. The present paper describes two methods that are suitable for the analysis of cable systems supported by elastic boundary structures, namely, the displacement method and a mixed method.

2. NOTATION

Indices.- Subscripts: the range of a lower case Latin subscript depends on the type of quantity to which the subscript is appended. For example, ξ_r is a generalized displacement component. The number of such components is n (the number of degrees of freedom of the structure), and the range of r is therefore 1 to n. Superscripts: lower case Greek superscripts denote element numbers and have the range 1 to N, where N is the number of elements. Lower case Latin superscripts enclosed in brackets denote iteration count. The letter T used as a superscript denotes transposition of vectors and matrices.

Summation convention. - This convention is used for subscripts but not for superscripts. This means that any term in which the same subscript appears twice stands for the sum of all such terms obtained by giving this subscript its complete range of values.

Vectors and matrices. - A column vector is denoted by a typical component enclosed in braces or by a lower case letter with a tilde written below the letter. A matrix is denoted by a typical component enclosed in square brackets or by a capital letter with a tilde written below the letter.

3. THE DISPLACEMENT METHOD

3.1 ELEMENT ANALYSIS

The structure is divided into a number of elements connected at nodes. The total number of elements is denoted by N. Consider one of the elements and let the generalized displacements of the nodes belonging to that element be denoted by x_i, $i = 1,2 \ldots,n_1$. It will be assumed that the x_i are defined with reference to a fixed global coordinate system.

In the displacement method we use approximate displacement functions for the elements determined by the generalized node displacements, so that the geometrical configuration of an element is completely and uniquely determined by the generalized node displacements x_i belonging to that element. The approximate displacements must also satisfy the condition that compatibility of node displacements for two neighbouring elements ensures displacement compatibility along the interelement boundary. Finally, the set of approximate displacement functions for an element must include all finite rigid-body displacements of that element.

Strain measures. The displacements throughout the element are determined by the generalized node displacements x_i. We now introduce another set of quantities, which determine the element displacements. Some of these, the strain measures e_k, determine the shape and thus the strains of the element, while the remaining quantities q_ℓ determine rigid-body displacements of the element. The relation between the two sets of quantities may be expressed by a transformation

$$x_i = x_i(e_1, e_2, \ldots, q_1, q_2, \ldots) , \qquad i = 1, 2, \ldots, n_1 . \tag{1}$$

with the following properties:

1. The transformation (1) is one-to-one and continuous. Denoting the number of e_k and q_ℓ-components by n_2 and n_3, respectively, we have $n_1 = n_2 + n_3$. Since the transformation is one-to-one we may solve (1) uniquely for e_k and q_ℓ, thus

$$e_k = e_k(x_1, x_2, \ldots, x_{n_1}), \; q_\ell = q_\ell(x_1, x_2, \ldots, x_{n_1}) . \tag{2}$$

2. The strain measures e_k are, in general, nonlinear functions of the x_i. The components e_k are the magnitudes of certain geometrical quantities (e.g. lengths, angles), which remain unchanged when the element is given a finite rigid-body displacement (i.e. the e_k are socalled objective invariants).

3. The quantities q_ℓ have the following properties: Suppose that the e_k are kept constant but that the q_ℓ vary. The corresponding change in the displacements of the element is then a finite rigid-body displacement.

In order to illustrate the concept of strain measures we can mention that for a straight pin-jointed rod there will be only one strain measure, which may be chosen as the elongation $(\ell-\ell_o)$ of the rod, ℓ being the current length of the rod and ℓ_o the length in the reference state. For a plane flexural element without shear deformations, there will be three strain measures (see Fig.1), and these may be chosen as the elongation of the chord-line and the angles ϕ_1 and ϕ_2, see also reference [9].

Fig.1

Strain energy. It is assumed that the element is elastic and that the strain energy U of the element is a function of the generalized node displacements x_i. As the strain energy is determined by the state of strain in the element, it can be shown that U depends only on the strain measures. Thus

$$U = U(e_1, e_2, \dots, e_{n_2}) .$$

As the e_k are functions of the x_i, see (2), we find for the partial derivatives of U with respect to x_i:

$$\frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial e_k} \frac{\partial e_k}{\partial x_i} , \qquad (3)$$

$$\frac{\partial^2 U}{\partial x_i \partial x_j} = \frac{\partial U}{\partial e_k} \frac{\partial^2 e_k}{\partial x_i \partial x_j} + \frac{\partial e_k}{\partial x_i} \frac{\partial^2 U}{\partial e_k \partial e_\ell} \frac{\partial e_\ell}{\partial x_j} , \qquad (4)$$

where the summation convention for subscripts has been used.

3.2 SYSTEM ANALYSIS

We number the elements 1, 2, ..., N, and in the following we shall use a superscript α to denote quantities associated with element No. α (e.g. x_i^α, e_k^α, U^α, etc.).

The generalized node displacements of all the nodes in the structure (defined with reference to the global coordinate system) are called ξ_r, $r = 1, 2, \dots, n$. There is a simple connection between the x_i^α and the ξ_r, since any node displacement x_i^α associated with element α is equal to one of the components ξ_r.

Equilibrium conditions. As far as the support conditions are concerned, we shall assume, for the sake of simplicity, that the structure is elastically supported, i.e. that the reactions at the support nodes are functions of the ξ_r-values belonging to these nodes, the generalized forces corresponding to the reactions being given by $-\partial U^S/\partial \xi_r$, where $U^S(\xi_1, \xi_2, \ldots)$ is a potential depending on the generalized displacements of the support nodes. As a simple example, the $\partial U^S/\partial \xi_r$ may be linear functions of the appropriate ξ_r-values, and the coefficients of the ξ_r in this linear expression may then be regarded as a kind of spring constants. It is well known that rigid supports are often approximated by linear elastic supports with very large spring constants for the purpose of convenient computer analysis.

The total elastic potential of the structure is the sum of the strain energy of the elements and the potential of the elastic supports, $\sum_{\alpha=1}^{N} U^\alpha + U^S$. The equilibrium conditions will be derived by means of the principle of virtual work. We express the fact that the work of the external loads corresponding to any compatible, infinitesimal displacement of the structure, equals the increase in elastic potential thus

$$\sum_{\alpha=1}^{N} dU^\alpha + dU^S = p_r \, d\xi_r \quad , \tag{5}$$

where the p_r are generalized forces corresponding to the external loads. (5) may be written in the form

$$\frac{\partial}{\partial \xi_r}\left(\sum_{\alpha=1}^{N} U^\alpha + U^S \right) d\xi_r = p_r \, d\xi_r \quad ,$$

and since U^α is a function of the generalized node displacements x_i^α belonging to element α, and the $d\xi_r$ are arbitrary, we obtain

$$\sum_{\alpha=1}^{N} \left(\frac{\partial U^\alpha}{\partial x_i^\alpha} \, \frac{\partial x_i^\alpha}{\partial \xi_r} \right) + \frac{\partial U^S}{\partial \xi_r} = p_r \quad , \tag{6}$$

where the partial derivatives $\partial x_i^\alpha/\partial \xi_r$ have the constant values zero or unity, see the previous remarks about the relation between x_i^α and ξ_r. The equations (6) express the equilibrium conditions for the structure. The unknowns in (6) are the generalized node displacements ξ_r, and the number of equations equals the number of unknowns.

Solution of nonlinear equations, the Newton-Raphson method. The governing equations (6) are generally nonlinear in the ξ_r. In the following, the Newton-Raphson method will be used for the solution, so that each step of the iteration process involves the solution of a linearized approximation to the nonlinear equations (6). The iteration process is specified by the following formula:

$$\left(\frac{\partial^2\left(\sum_{\alpha=1}^{N} U^\alpha + U^S\right)}{\partial\xi_r\,\partial\xi_s}\right)^{(k)} d\xi_s^{(k+1)} = \left(p_r - \frac{\partial}{\partial\xi_r}\left(\sum_{\alpha=1}^{N} U^\alpha + U^S\right)\right)^{(k)} , \qquad (7)$$

where

$$\frac{\partial U^\alpha}{\partial\xi_r} = \frac{\partial U^\alpha}{\partial x_i^\alpha}\,\frac{\partial x_i^\alpha}{\partial\xi_r} , \quad \frac{\partial^2 U^\alpha}{\partial\xi_r\,\partial\xi_s} = \frac{\partial x_i^\alpha}{\partial\xi_r}\,\frac{\partial^2 U^\alpha}{\partial x_i^\alpha\,\partial x_j^\alpha}\,\frac{\partial x_j^\alpha}{\partial\xi_s} .$$

In this formula, the k'th iterates $\xi_r^{(k)}$ are used to evaluate the residuals on the right-hand side and the coefficients of $d\xi_s^{(k+1)}$ on the left-hand side. Having solved the set of linear equations (7), the (k+1)'st iterate is given by

$$\xi_r^{(k+1)} = \xi_r^{(k)} + d\xi_r^{(k+1)} . \qquad (8)$$

The symmetric matrix of coefficients in (7) is called _the stiffness matrix of the structure_ and is a sum of contributions from the elements and the elastic supports. The $\left[\frac{\partial^2 U^\alpha}{\partial x_i^\alpha\,\partial x_j^\alpha}\right]$ form _the stiffness matrix of the element_ (for given α); this matrix may be written as a sum of the geometrical and the elastic stiffness matrix, see (4).

Properties of the theory. The introduction of strain measures that are objective invariants is an important feature of the present theory. By taking the strain energy of an element to be a function of the strain measures, we satisfy the important condition that a finite rigid-body displacement should not change the state of stress in an element. Approximations may be introduced in a rational manner. We may use exact expressions for the strain measures as functions of the node displacements but introduce approximate expressions for the strain energy functions U^α of the elements. As a further approximation, we may use approximate expressions for the strain measures that are not exact objective invariants. A more detailed derivation of the theory is given in the writer's paper [9], see also Argyris & Scharpf [1].

3.3 STIFFNESS MATRICES

In the following we shall consider structures made up of simple tension elements and flexural elements. It is not possible within the scope of the present paper to present detailed expressions for constitutive relations and stiffness matrices of these elements. We shall therefore confine ourselves to a few remarks on this subject and give some relevant references.

Simple tension element. This is a straight, pin-jointed rod, which is perfectly flexible. The global components of the node displacements are used as generalized displacements (6 components for the element), and the elongation of the rod is used as strain measure. The element is assumed to be linearly elastic. The relevant formulae for the strain energy of the element, the internal force as a function of the elongation, and the stiffness matrix of the element, may be found in reference [9].

Straight flexural element in space. The formulae for the flexural element that are used in the following numerical example are based on a simplified quasi-linear theory in which the rotations are assumed to be small but in which the effects of the displacements (the changed geometry) are taken into account when formulating the equations of equilibrium. The global components of displacements and rotations of the nodes are used as generalized displacements (12 components for the element). There are 6 strain measures, which include the elongation of the chord-line and certain angles associated with the bending and torsion of the element. These strain measures are approximate in the sense that they are not exact objective invariants. Formulae for the strain measures as functions of the generalized displacements, the strain energy of the element, the generalized internal forces as functions of the generalized displacements, and the stiffness matrix of the element, are given in reference [9].

4. A MIXED METHOD

We consider a cable net supported by an elastic boundary structure in the form of a ring beam, see Fig. 2. In the simplified system used for the analysis, the cable net is represented by

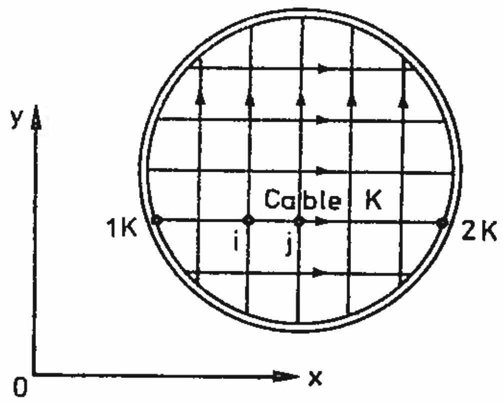


Fig. 2

a series of simple tension elements and the ring beam is represented by a series of straight flexural elements. A global cartesian coordinate system XYZ, with vertical Z-axis, is introduced. It is assumed that the cables are in vertical planes in the initial state and that the horizontal projections of the cables in this state are straight lines parallel to the X- and Y-axes. The cables are assumed to be shallow, the elastic cable stiffness EA is assumed to be constant along a cable, and the external loads acting on the cable net are assumed to be vertical forces at the nodes (the latter assumption is not strictly necessary, but it makes the resulting formulae simpler).

The mixed method may be derived from the previous results by the introduction of certain approximations based on the following assumptions:

1. The strains of the cable elements are small.
2. The horizontal components of the displacement of a cable net node are small compared with the vertical component.

The following notation is introduced in connection with the present method:

Difference operator Δ. Let a quantity f_i be defined at all cable net nodes i. For a directed element ij of the cable net we define (see also Fig.3):

$$\Delta f_{ij} = f_j - f_i$$

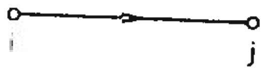


Fig.3

We shall occasionally omit the subscripts and just write Δf, it being understood that the difference is to be formed corresponding to a given orientation of the element by subtracting the f-value of the initial node from the f-value of the terminal node.
Position vector of node i, initial state: $\underset{\sim}{r}_i = [x_i\ y_i\ z_i]^T$.
Displacement vector of node i (displacement measured from initial state): $\underset{\sim}{v}_i = [u_i\ v_i\ w_i]^T$

$$\Delta \underset{\sim}{r}_{ij} = \underset{\sim}{r}_j - \underset{\sim}{r}_i = [\Delta x\ \Delta y\ \Delta z]^T_{ij} ,$$

$$\Delta \underset{\sim}{v}_{ij} = \underset{\sim}{v}_j - \underset{\sim}{v}_i = [\Delta u\ \Delta v\ \Delta w]^T_{ij} .$$

In the following we shall occasionally omit the subscripts i,j (which denote node numbers in the cable net).
The length of element ij is denoted by:
Initial state: $\ell^* = (\Delta \underset{\sim}{r}^T\ \Delta \underset{\sim}{r})^{\frac{1}{2}}$,
Current state: $\ell = [(\Delta \underset{\sim}{r} + \Delta \underset{\sim}{v})^T\ (\Delta \underset{\sim}{r} + \Delta \underset{\sim}{v})]^{\frac{1}{2}}$.

The cables are numbered consecutively, and positive directions are introduced on the cables corresponding to the positive X- and Y-directions, see Fig.2. Consider now an element ij on cable No. K corresponding to the X-direction, see Fig.2. Using assumptions 1. and 2. and neglecting small quantities, the following approximate expression is obtained for the elongation of the element (measured from the initial state):

$$\ell - \ell^* = \frac{1}{\ell^*}\ (\Delta x\ \Delta u + \Delta z\ \Delta w + \frac{1}{2}\Delta w^2) \quad . \tag{9}$$

In this and the following formula, the differences corresponding to the Δ-terms are to be evaluated for an element orientation coinciding with the orientation of cable K. As the external loads consist of vertical forces, it follows from our assumptions that the horizontal component of the internal force in cable elements is constant along a cable, and we obtain the approximate formula:

$$\left[(H-H^*)\frac{\Lambda L}{EA}\right]_K = u_{2K} - u_{1k} + \sum_{\text{cable } K} \left\{ \frac{\Delta z + \Delta w/2}{\Delta x}\ \Delta w \right\} , \tag{10}$$

where the summation includes all the elements of cable K, H and H^* are the constant horizontal components of the cable force in the current state and the initial state, respectively, u_{1K} and u_{2K} are the X-components of the displacements of the terminal nodes of cable K, see Fig.2, $L = x_{2K} - x_{1K}$ is the length of the horizontal projection of cable K, and the constant Λ is given by

$$\Lambda = \frac{1}{L} \sum_{\text{cable}} \left(\frac{\ell^{*3}}{\Delta x^2} \right) \quad . \tag{11}$$

In order to obtain formulae for the cables corresponding to the Y-direction, x and u in formulae (9) and (10) are replaced by y and v, respectively.

The strain energy of a cable element ij may be written in the form

$$U = \frac{1}{2} \frac{H^2}{EA} \frac{\ell^{*3}}{\Delta x^2} \quad , \tag{12}$$

where H is given in terms of the displacements by (10). It will be seen that U depends on the vertical displacements of the cable nodes and on the horizontal displacements of the terminal nodes of cable K, i.e. we have eliminated the horizontal displacements of the intermediate nodes.

If we now form the equations of equilibrium according to (6), using the above approximate expression for the strain energy of the cable elements, the following equations are obtained (the partial derivatives of the function Φ, which appear in these equations, should be disregarded at a first reading):

Vertical projection at internal cable net nodes.

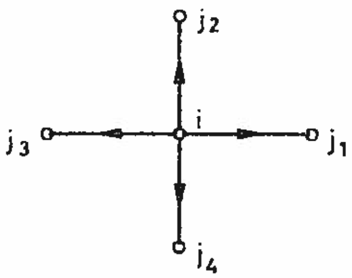


Fig.4

$$\frac{\partial \Phi}{\partial w_i} = -\sum_{\nu=1}^{4} \left\{ H \frac{\Delta z + \Delta w}{d} \right\}_{ij_\nu} - p_i = 0 \quad , \tag{13a}$$

where the subscripts ij_ν indicate that the term enclosed in braces is to be evaluated for the directed element ij_ν, see Fig.4, p_i is the Z-component of the external load acting at node i, and $d = |\Delta x|$ or $d = |\Delta y|$ depending on whether the projected element is parallel to the X- or Y-axis, respectively.

Equations of equilibrium of ring beam. The components of displacements and rotations of the ring beam nodes are denoted by ξ_r, and the strain energy of ring beam element No. α is denoted by $U^{\alpha r}$, which is a function of the ξ_r. The potential of the elastic supports is denoted by U^S, and this is also a function of the ξ_r. We shall assume that the terminal nodes of the cables coincide with ring beam nodes, so that the displacements of nodes 1K and 2K appearing in (10) are, in fact, contained among the ξ_r. The 6 equilibrium conditions for node g on the ring beam have the form (see also Fig.5)

$$\left(\frac{\partial\phi}{\partial\xi_r}\right)_{\text{node } g} =$$

$$= \left[\frac{\partial}{\partial\xi_r}\left(\sum_\alpha U^\alpha + U^S\right)\right]_{\text{node } g} - \begin{bmatrix} \pm H_K + p_x \\ \pm H_J + p_y \\ \left(H\frac{\Delta z + \Delta w}{|\Delta x|}\right)_{gj} + \left(H\frac{\Delta z + \Delta w}{|\Delta y|}\right)_{gi} + p_z \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (13b)$$

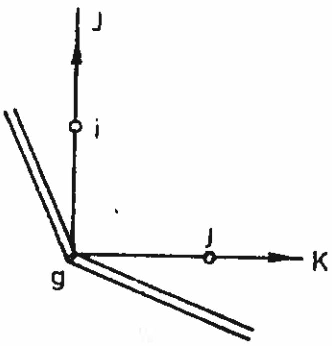


Fig.5

where

$$\left\{\frac{\partial}{\partial\xi_r}\right\}_{\text{node } g} = \left[\frac{\partial}{\partial u}\ \frac{\partial}{\partial v}\ \frac{\partial}{\partial w}\ \frac{\partial}{\partial\theta_x}\ \frac{\partial}{\partial\theta_y}\ \frac{\partial}{\partial\theta_z}\right]^T_{\text{node } g} ,$$

θ_x, θ_y, θ_z are the components of the rotation of joint g , and p_x, p_y, p_z are the components of the external load acting at node g. The plus sign in the H-terms should be used when node g is the initial node of the cable in question, and the minus sign, when node g is the terminal node of the cable.

Conditions of cable compatibility. We also have equation (10) which determines the horizontal component of the cable force as a function of the displacements,

$$\frac{\partial\Phi}{\partial H_K} = u_{2K} - u_{1K} + \sum_{\text{cable } K}\left\{\frac{\Delta z + \frac{1}{2}\Delta w}{\Delta x}\Delta w\right\} - \left[\frac{\Lambda L}{EA}(H-H^*)\right]_K = 0 \quad . \ (13c)$$

Equations (13a, b, c) form a system of nonlinear equations for the determination of the unknown node displacements w_i and ξ_r and the horizontal components of the cable forces H_K . These are the governing equations of the present mixed method.

Let us introduce the following mixed potential function:

$$\phi = \sum_\alpha U^\alpha(\xi_r) + U^S(\xi_r) + \sum_{\substack{\text{all cable}\\\text{elements}}}\left\{H\frac{\Delta z + \frac{1}{2}\Delta w}{d}\Delta w\right\} -$$

$$- \sum_K\left\{\frac{1}{2}\frac{\Lambda L}{EA}(H-H^*)^2 - H(u_{2K}-u_{1K})\right\}_K - \sum_i p_i w_i - \sum_r p_r\xi_r \quad , \qquad (14)$$

where it has been assumed that the components of the external loads p_i and p_r are independent of the displacements. It will now be seen that the governing equations (13a, b, c) can be obtained by putting the 1st derivatives of Φ with respect to the variables w_i, ξ_r and H_K equal to zero, as has already been shown in the equations.

The Newton-Raphson method will again be used for the solution so that we have the iteration formula

$$\left(\frac{\partial^2\Phi}{\partial\zeta_r\ \partial\zeta_s}\right)^{(k)} d\zeta_s^{(k+1)} = -\left(\frac{\partial\Phi}{\partial\zeta_r}\right)^{(k)}, \qquad (15)$$

where the complete vector of unknowns is denoted by $\{\zeta_r\}$. (15) corresponds to the linearized form of equations (13a, b, c). An additional simplification will now be introduced in connection with the linearization. The vertical displacements w_g of the ring beam nodes are generally small compared with the vertical displacements of the cable net nodes. Therefore, when calculating the left-hand side of (15), we put $dw_g = 0$ in the part of $(\partial^2\Phi/\partial\zeta_r\ \partial\zeta_s)d\zeta_s$ that is due to the term $\Sigma\{H\Delta w(\Delta z + \frac{1}{2}\Delta w)/d\}$ in the expression for Φ. In this way the following linearized equations are obtained (in which the index denoting iteration count has been omitted),

$$-\sum_{\nu=1}^{4}\left\{H\,\frac{\Delta(dw)}{d} + dH\,\frac{\Delta z + \Delta w}{d}\right\}_{ij_\nu} = \left\{-\frac{\partial\Phi}{\partial w_i}\right\}, \qquad (16a)$$

$$\left(\frac{\partial^2\left(\sum_\alpha U^\alpha + U^S\right)}{\partial\xi_r\ \partial\xi_s}\,d\xi_s\right)_{\text{node } g} - \begin{bmatrix} \pm dH_K \\ \pm dH_J \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \left\{-\frac{\partial\Phi}{\partial\xi_r}\right\}_{\text{node } g}, \qquad (16b)$$

$$du_{2K} - du_{1K} + \sum_{\text{cable } K}\left\{\frac{\Delta z + \Delta w}{d}\,\Delta(dw)\right\} - \left(\frac{\Delta L}{EA}\,dH\right)_K = -\frac{\partial\Phi}{\partial H_K}. \qquad (16c)$$

These equations may be written in matrix form as follows:

$$\begin{bmatrix} \underset{\sim}{D}(\underset{\sim}{H}) & \underset{\sim}{0} & \underset{\sim}{Z}(\underset{\sim}{w}) \\ \underset{\sim}{0}^T & \underset{\sim}{K}(\underset{\sim}{\xi}) & \underset{\sim}{Y} \\ \underset{\sim}{Z}^T(\underset{\sim}{w}) & \underset{\sim}{Y}^T & \underset{\sim}{F} \end{bmatrix} \begin{bmatrix} d\underset{\sim}{w} \\ d\underset{\sim}{\xi} \\ d\underset{\sim}{H} \end{bmatrix} = \begin{bmatrix} -\left\{\frac{\partial\Phi}{\partial w_i}\right\} \\ -\left\{\frac{\partial\Phi}{\partial\xi_r}\right\} \\ -\left\{\frac{\partial\Phi}{\partial H_K}\right\} \end{bmatrix} \qquad (17)$$

The matrix of coefficients is symmetrical because the components of this matrix are 2nd derivatives of the potential ϕ . $\underset{\sim}{0}$ denotes a nullmatrix, and the dependence of the submatrices on the unknowns has also been indicated. The submatrices are obtained as a sum of contributions as follows:

Matrix $\underset{\sim}{D}$: A tension element connecting two internal cable net nodes i and j contributes:

Column to be multiplied by: dw_i dw_j

To be added to equation corresponding to cable net node No.
$$\begin{cases} i: \\ j: \end{cases} \quad \frac{H}{d}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} .$$

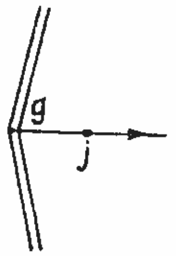


Fig.6

A tension element connected to node g on the ring beam contributes only the term dw_j H/d in the equation corresponding to cable net node j , see Fig.6.

Matrix $\underset{\sim}{Z}$: A tension element ij on cable K contributes:

In equation corresponding to cable net node No.
$$\begin{cases} i: & \frac{1}{d}[(z+w)_i-(z+w)_j]dH_K \;, \\ j: & \frac{1}{d}[(z+w)_j-(z+w)_i]dH_K \;. \end{cases}$$

Matrix $\underset{\sim}{Y}$: For a cable K corresponding to X-direction:

At node 1K, projection in X-direction: $-1 \times dH_K$;
At node 2K, projection in X-direction: $+1 \times dH_K$.

Similar expressions hold good for the Y-direction.

Matrix $\underset{\sim}{K}$: This is the stiffness matrix of the ring beam assembled from the stiffness matrices of the flexural elements, see Section 3.3, and including contributions from elastic supports.

Matrix $\underset{\sim}{F}$: This is a diagonal matrix, the diagonal component corresponding to cable K being given by $-(\Lambda L/(EA))_K$.

Comments on mixed method. A mixed method somewhat similar to the above method was presented by Kärrholm & Samuelsson [5]. The main differences between their method and that of the writer are as follows: Kärrholm & Samuelsson's method involves a preliminary calculation in which the flexibility matrix of the ring beam supported by columns has to be established by means of a finite element analysis (displacement method) of the ring beam structure. Also, the iteration method used for solving the nonlinear equations is not the systematic Newton-Raphson method but a special iteration procedure in which values of the resultant horizontal components of the cable forces must be estimated in advance in order to start the iteration. It would also appear that a mixed method was used for the design of the structure described by Ferretti & Zingali in [4], although the description of their method of analysis given in [4] is so brief that it is not possible to compare it with the present method.

5. NUMERICAL EXAMPLE

We now consider a structure of the type described in Section 4, i.e. a cable net supported by a ring beam. The centreline of the ring beam is a space curve determined by the intersection between the saddle-shaped roof surface and a vertical circular cylinder (diameter = 104 m). The ring beam is assumed to be a reinforced concrete structure, the cross-section of the ring beam being rectangular as shown in Fig.7. The cross-section of the beam twists as we move along the perimeter, the longer side of the rectangle being parallel to the tangent to the roof surface in the direction normal to the boundary.

In the analysis, the cable net is represented by 9 hanging cables (corresponding to the X-direction) and 9 bracing cables (corresponding to the Y-direction). The ring beam is made up of 28 straight flexural elements and is supported by vertical columns at the nodes between the flexural elements (the effect of the columns is represented by elastic supports which produce vertical reactions). Nodes 1 and 3 on the ring beam (see Fig.7) are prevented from moving in the Y-direction, and node 2 is prevented from moving in the X-direction (these horizontal constraints are accounted for in the analysis by means of elastic supports with very large spring constants).

The stiffness parameters of the cable net have the following values:

$E = 160000\ MN/m^2$,

Hanging cables: EA = 832 MN per cable
Bracing cables: EA = 832 MN per cable

The flexural elements making up the ring beam are assumed to be elastic with a modulus of elasticity of $E_c = 2 \times 10^4\ MN/m^2$. The cross-sectional constants have the values shown in Fig.7.

Initial state. The geometry of the cable net is determined as the equilibrium configuration corresponding to the following loading: Prestress in the cables, weight of roof cladding and cables on the net, and weight of the ring beam, and it is also specified that the cables should be in vertical planes in this state, which is called the initial state. The method used for determining the initial state is presented in reference [9]. It is found that the cable net nodes are located very nearly on a hyperbolic paraboloid in the initial state, and the horizontal components of the cable forces in this state are given by:

Hanging cables: 2.60 MN per cable
Bracing cables: 2.60 MN per cable

Loading. The cable net is calculated for the following vertical load on the roof surface (measured per unit area of horizontal projection in the initial state):

Dead load (weight of roof cladding and cables): $600\ N/m^2$
Live load (snow): $750\ N/m^2$

The two loading cases referred to in the following are given by:

Loading case 1: Uniformly distributed dead load plus live load on the whole roof.

Loading case 2: Dead load on the whole roof plus snow load on half the roof (i.e. for y>0).

The displacement method. A FORTRAN program was developed for the analysis of structures of the present type by means of the displacement method as described in Section 3. The unknowns are the displacements and rotations of the ring beam nodes and the displacements of the cable net nodes (a total of 375 unknowns). The results of the analysis of the two loading cases are presented in Figs. 7 and 8. It was found that four or five iterations per loading case were required in order to obtain the desired accuracy ($\max|d\xi_r| / \max|\xi_s| < 10^{-3}$ should be satisfied for both the set of cable net displacements and for the set of ring beam displacements). The calculations were performed on an IBM 370 computer using single precision arithmetic. The symmetry conditions of the results were satisfied to an accuracy of four to five significant figures, indicating that the linear equations solved in the Newton-Raphson iteration are quite well-behaved.

The mixed method. A FORTRAN program was also developed for the analysis of this type of structure by means of the mixed method. The total number of unknowns is in this case 255 . The results of the analysis are not presented here as they agree closely with the results obtained with the displacement method. Three or four iterations per loading case were required in order to obtain the desired accuracy.

Comments. It will be seen that the behaviour of the present rather flexible structure (cable net supported by elastic ring beam) differs considerably from that of a cable net with rigid supports. Consider the case of a uniformly distributed load on the cable net (loading case 1). In the present example (elastic ring beam), the forces in both the hanging and the bracing cables increase, and the resultant forces in the two cable systems are almost the same, so that the moments in the ring beam are fairly small. This is in contrast to a cable net with rigid supports, in which the forces in the bracing cables are significantly reduced for a loading of this type.

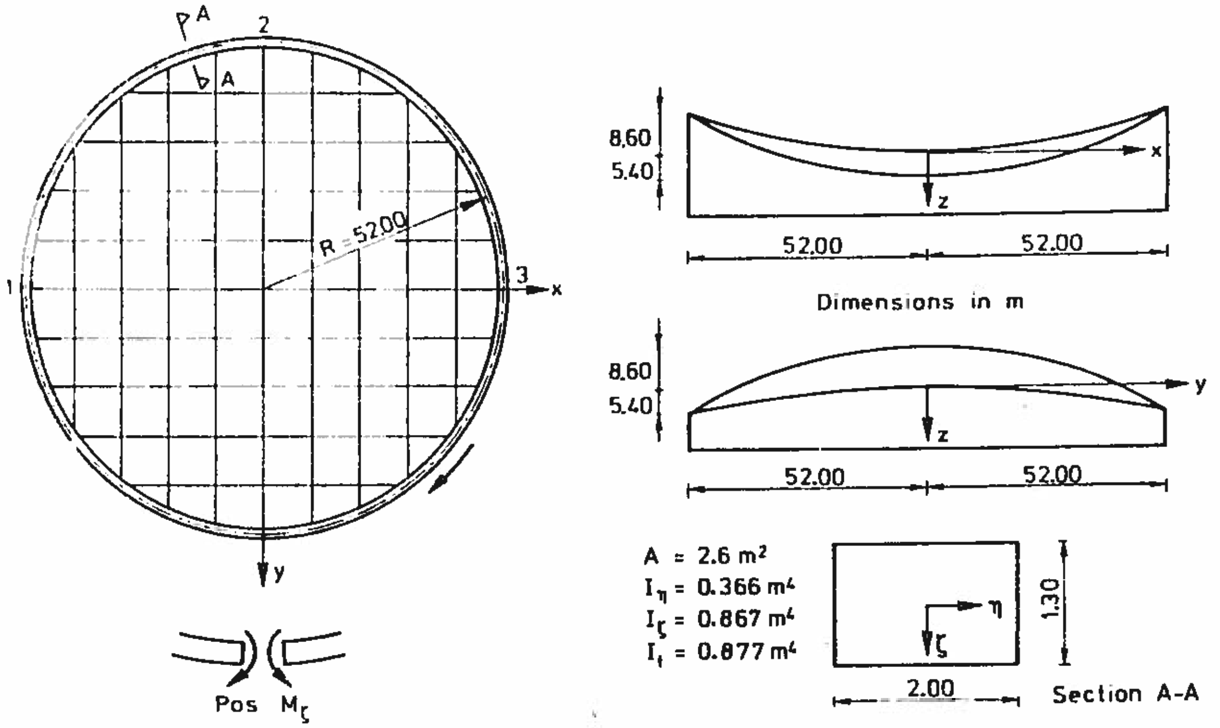
6. CONCLUSIONS

Two methods have been presented for the analysis of cable systems supported by elastic boundary structures: 1. The displacement method applied to the complete structure (cable system plus boundary structure) and 2. a mixed method, in which the unknowns are the generalized node displacements of the boundary structure, the vertical components of the cable net node displacements, and the horizontal components of the cable forces. In both cases the Newton-Raphson method was used for the solution of the governing nonlinear equations.

Numerical examples show that both methods can be used successfully for the analysis of this type of structure. No difficulties were experienced with the convergence of the methods even when the boundary structure was quite flexible. Comparing the two methods, the following conclusions can be drawn: The main advantage of the displacement method is its generality - the method is not restricted by the simplifying assumptions and approximations inherent in the mixed method (shallow cables in vertical planes etc.). The mixed method, on the other hand, has the advantage that, for a given structure, the number of unknowns appearing in the equations of the mixed method is considerably smaller than the number of unknowns in the equations of the displacement method, with correspondingly reduced requirements to computer time and storage capacity.

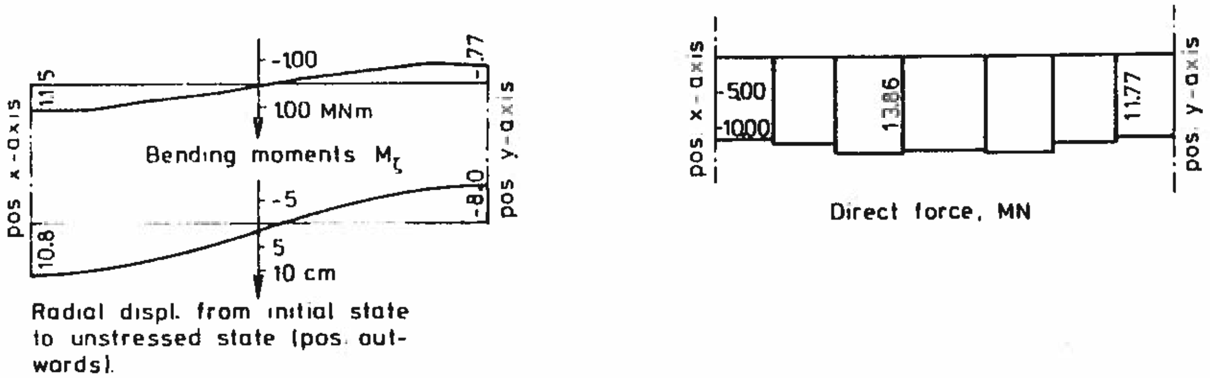
NUMERICAL EXAMPLE

GEOMETRY



INITIAL STATE

RESULTS FOR RING BEAM



LOADING CASE 1

RESULTS FOR NET

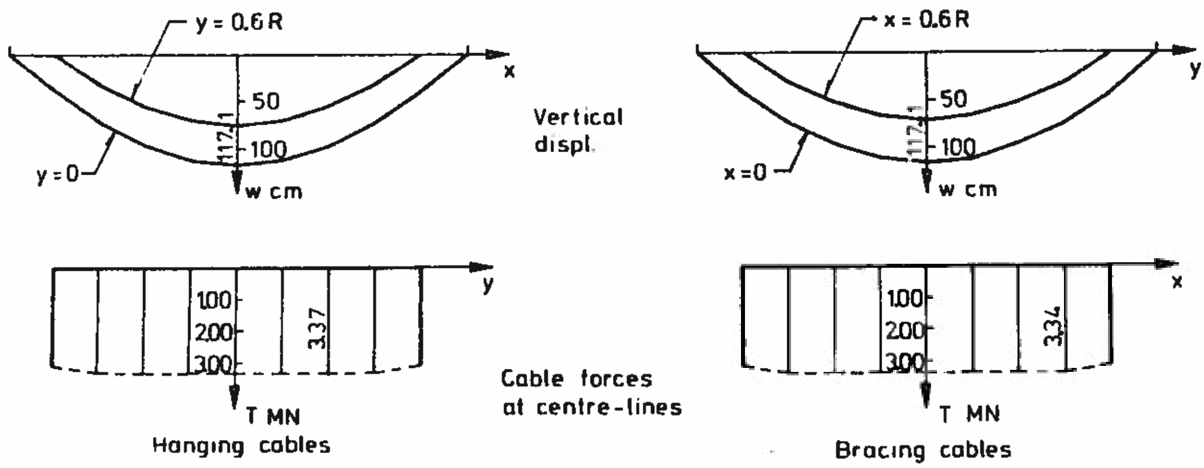


Fig. 7

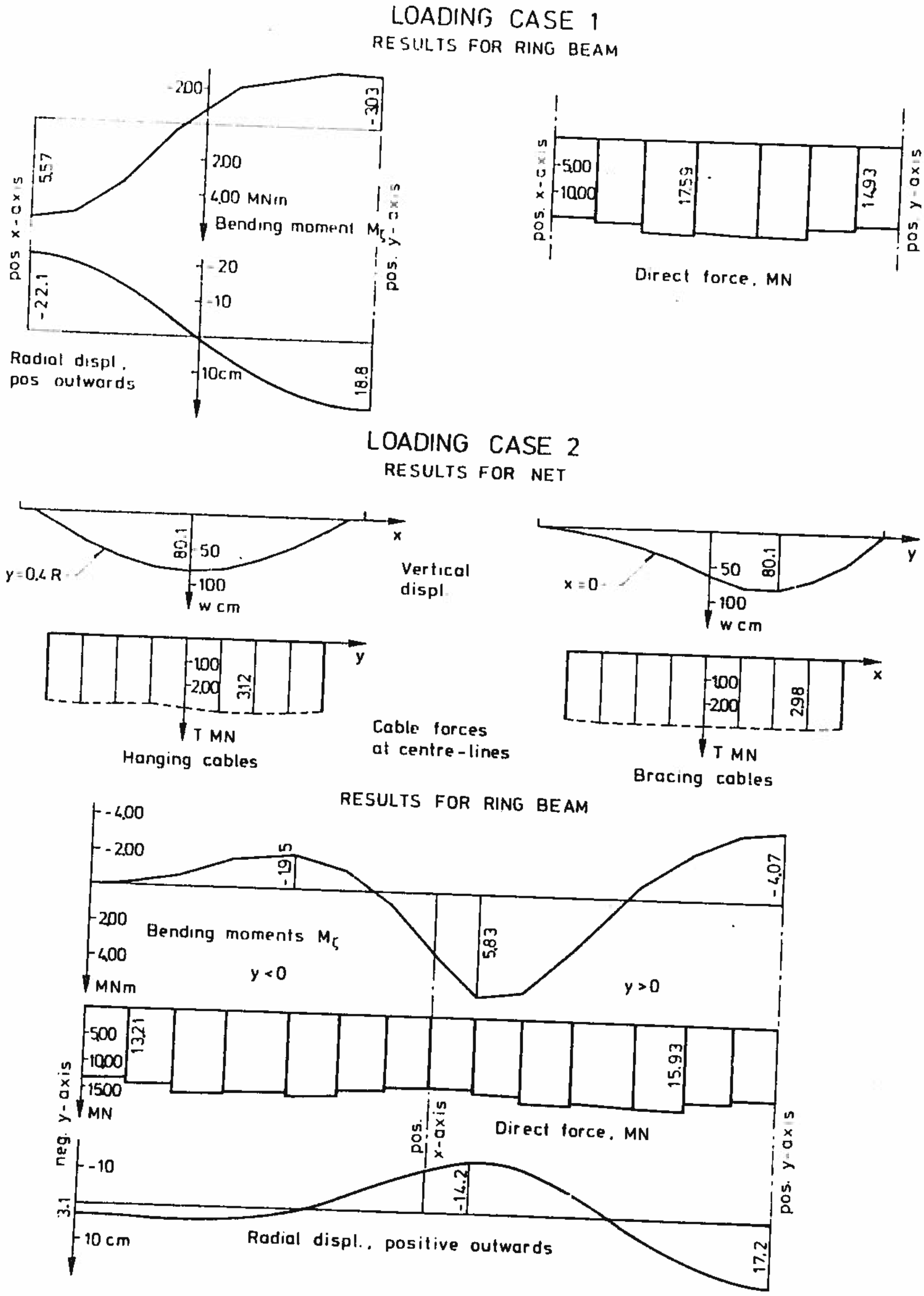

LOADING CASE 1
RESULTS FOR RING BEAM
-200
-3.03
5.57
200
4.00 MNm
Bending moment Mζ
pos x-axis
pos. y-axis
-20
-10
-22.1
Radial displ., pos outwards
10cm
18.8
pos. x-axis
5.00
10.00
17.59
14.93
pos. y-axis
Direct force, MN
LOADING CASE 2
RESULTS FOR NET
x
80.1
50
y=0.4 R
100
w cm
Vertical displ.
y
x=0
50
80.1
100
w cm
y
1.00
2.00
3.12
T MN
Hanging cables
Cable forces at centre-lines
x
1.00
2.00
2.98
T MN
Bracing cables
RESULTS FOR RING BEAM
-4.00
-2.00
-1.95
-4.07
2.00
4.00
MNm
Bending moments Mζ
y < 0
5.83
y > 0
5.00
10.00
15.00
MN
13.21
15.93
neg. y-axis
pos. x-axis
Direct force, MN
-10
3.1
10 cm
-14.2
Radial displ., positive outwards
pos. y-axis
17.2

Fig. 8

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