AFDELINGEN FOR

BÆRENDE KONSTRUKTIONER

DANMARKS TEKNISKE HØJSKOLE

STRUCTURAL RESEARCH LABORATORY

TECHNICAL UNIVERSITY OF DENMARK

M. P. Nielsen and M. W. Bræstrup

PLASTIC SHEAR STRENGTH OF REINFORCED CONCRETE BEAMS

RAPPORT NR. R 73 1976

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Særtryk af
BYGNINGSSTATISKE MEDDELELSER
VOL. 46 . NO. 3 . 1975

PLASTIC SHEAR STRENGTH OF REINFORCED CONCRETE BEAMS

M. P. NIELSEN★ M. W. BRÆSTRUP★★

1. INTRODUCTION

The ultimate strength of reinforced concrete beams subjected to shear has received considerable interest during recent years and the literature on the subject is almost overwhelming. Most papers, however, report specific test series and supply formulas to accomodate the results obtained. Attempts to predict the shear resistance on a rational analysis of the behaviour of the beam materials are rather few.

Traditionally the shear strength of reinforced concrete beams has been calculated by means of the Mörsch truss analogy. Since experience shows that the load-carrying capacity is seriously underestimated, attempts have been made to improve the formula by adding various terms. Thus most codes of practice allow a »contribution from the concrete» independent of the shear reinforcement. Many other and more sophisticated additive formulas may be found in the literature.

Another trend is represented by the potential formulas which are derived by expressing the shear strength as a power product of the relevant parameters. The unknown exponents are then found by correlation with tests.

By combining the two kinds of formulas it is possible to arrive at expressions that agree remarkably well with any given test series.

★ Professor, dr.techn., Structural Research Laboratory, Technical University of Denmark.

★★ Civil engineer, lic.techn., Structural Research Laboratory, Technical University of Denmark.

Only a few authors have sought a solution to the shear problem using the mathematical theory of plasticity. This fact is puzzling considering the remarkable results obtained by plastic analysis of slabs, discs, and beams in bending. In deriving a rational estimate of the shear strength of a reinforced concrete beam, the crux of the matter is to establish an expression for the inclination φ of the uniaxial web compression. Assuming plastic properties of the beam materials Nielsen [1] determined φ by the requirement that the total amount of reinforcement be at a minimum. In a subsequent discussion Nielsen [2] established the yield load when web crushing is critical.

Earlier attempts to use the theory of plasticity (Borishansky & Gvozdev [3], Sigalov & Strongin [4], pp. 79 ff.) are based on an upper-bound technique where the end of the beam is supposed to rotate about a plastic hinge at the head of the crack (fig. 1). A similar mechanism forms the basis of the shear compression theories of Walther [5], Regan & Placas [6], and others.

The mechanism of fig. 1, however, is contradicted by experiments which show no evidence of yielding of the main reinforcement crossing the crack. Hence, neglecting the elastic elongation of the reinforcement, we conclude that the centre of rotation - if any - is not at the head of the crack, but at some position at the level of the main reinforcement. Thus the deformation is not a simple opening of the crack but involves tangential shearing as well.

The »interface shear transfer» is taken into account in the report of the joint ASCE-ACI Task Committee 426 [7]. The existence of shear forces in the diagonal crack has indeed been demonstrated by various experiments, e.g. by Taylor [8], who prefers the label »aggregate interlock». Here we shall simply regard these forces as arising from the resistance of the concrete against shear deformations, as displayed by the failure condition.

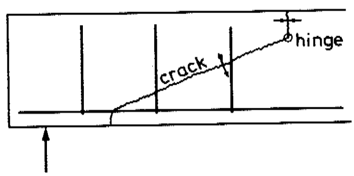


Fig. 1. Failure mechanism according to Borishansky & Gvozdev [3], Sigalov & Strongin [4], Walther [5], Regan & Placas [6], and others.

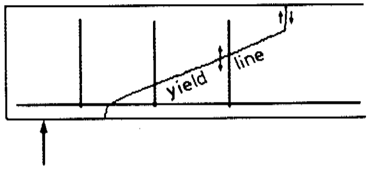


Fig. 2. Failure mechanism adopted in the present paper.

In the present paper we shall establish an upper bound, using a mechanism with no rotation at all, i.e. we assume the relative displacement rate of the two sides of the crack to be purely vertical at failure (fig. 2). We shall demonstrate that the corresponding upper bound for the load-carrying capacity is identical to the lower bound furnished by Nielsen's web crushing criterion.

2. PHYSICAL MODEL

We assume that the beam is in a state of plane stress, i.e. we regard the beam as a two-dimensional body. We consider the part of a rectangular beam of width b occupying the region $0 \leqslant x \leqslant a$ and $0 \leqslant y \leqslant h$, where a is the shear span and h is the depth of the beam (fig. 3).

The section $x = 0$ corresponds to a point of inflexion and $x = a$ is a point of maximum moment. At an arbitrary section $x = x$ the beam is subjected to a moment M and a shear force V.

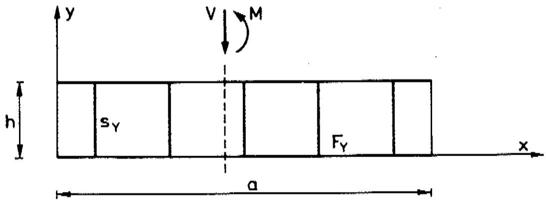


Fig. 3. Two-dimensional beam model.

We assume that the beam has a tension stringer at y = 0 and a compression stringer at y = h. Both stringers are perfectly plastic, the yield force of the tension stringer (main reinforcement) being F_y, assumed to be constant throughout the shear span. The shear reinforcement consists of perfectly plastic, vertical stirrups whose constant spacing is sufficiently small so they can be characterized by the equivalent yield stress s_y per unit area perpendicular to the stirrup direction (the parameter s_y is often denoted rf_y). The reinforcement as well as the stringers are unable to resist deformations perpendicular to their direction.

The concrete occupying the region $0 < y < h$ is supposed to be perfectly plastic with a square yield locus corresponding to the compressive yield stress σ_c and the tensile yield stress zero (fig. 4).

As an example of the application of the square yield locus and the theory of plasticity to concrete in plane stress, let us consider the failure of concrete under a uniaxial, compressive stress σ_b (fig. 5).

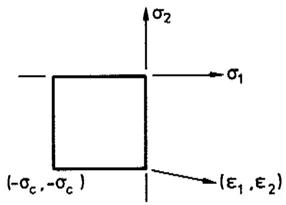


Fig. 4. Yield locus for concrete in plane stress.

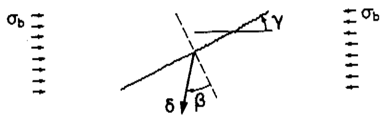


Fig. 5. Failure of concrete under uniaxial compression.

A lower bound for the failure stress is obviously

$$\sigma_b = \sigma_c$$

An upper bound is found assuming a straight yield line forming the angle γ with the direction of compression. Let δ be the relative displacement rate across the yield line and let it be inclined at the angle β to the yield line normal (fig. 5). The rate of internal work (plastic dissipation) per unit length, corresponding to the square yield locus is then (see e.g. Nielsen [9], pp. 69 ff.):

$$W_I = \frac{1}{2} \sigma_c \delta (1 - \cos \beta) \tag{1}$$

The rate of external work is

$$W_E = \sigma_b \delta \sin (\beta - \gamma) \sin \gamma$$

Equating the rates of internal and external work we find an upper bound for the failure stress:

$$\sigma_b = \frac{(1 - \cos \beta)\sigma_c}{2\sin(\beta - \gamma) \sin \gamma}$$

The minimum upper bound is found from the conditions

$$\frac{\partial \sigma_b}{\partial \beta} = 0 \qquad \text{and} \qquad \frac{\partial \sigma_b}{\partial \gamma} = 0$$

Both equations yield the result:

$$\beta = 2\gamma \tag{2}$$

which gives

$$\sigma_b = \sigma_c$$

Thus the assumed mechanism gives the correct solution and we obtain the relation (2) between the directions of the yield line and the relative displacement rate under uniaxial compression.

We have used the term »yield line» to designate the kinematic discontinuity, because the term »crack» may be somewhat confusing. Traditionally, cracks in concrete are conceived as brittle fractures caused by tensile stresses. Thus they follow the direction of principal compressive stress and there is no shear transfer. This requires that the relative displacement rate be perpendicular to the crack ($\beta = 0$). On the other hand, it is customary to speak of »shear cracks» in connection with shear in beams, where the direction of the displacement rate is governed by the main reinforcement. In order to avoid ambiguity we shall use the term »yield line» in such cases. Thus a yield line is a curve where the deformations are located. It is a mathematical idealization of a narrow region with many criss-crossing cracks and crushing zones. In the case of plane stress, the relative displacement rate may be at any angle to the yield line, which in general does not follow a principal stress trajectory.

3. *CONCENTRATED LOADING*

3.1. *Lower-Bound Solution*

Let us consider a traditional shear test, i.e. a simply supported beam subjected to two concentrated loads P applied symmetrically at the distance a from the supports (fig. 6).

We derive a lower bound for the load P by assuming a statically admissible stress distribution. Denoting the elements of the stress tensor in the x,y-system by σ_x, σ_y, τ_{xy} and the stringer force by N_x, we assume the state of stress at an arbitrary beam section x = x given by:

$$N_x = -C \qquad \text{for } y = h \qquad (3a)$$

$$\left.\begin{aligned} \sigma_x &= -\sigma_b \cos^2\varphi \\ \sigma_y &= -\sigma_b \sin^2\varphi + s \\ \tau_{xy} &= -\sigma_b \cos\varphi \sin\varphi \end{aligned}\right\} \qquad \text{for } 0 < y < h \qquad (3b)$$

$$N_x = F \qquad \text{for } y = 0 \qquad (3c)$$

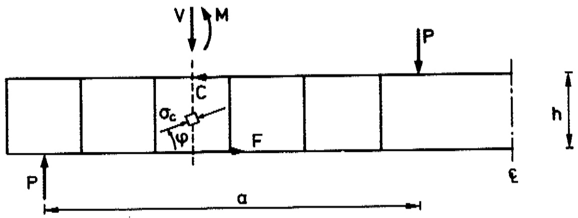


Fig. 6. Stress distribution for simple beam with symmetrical point loads.

Here C and F are the compressive and tensile stringer forces, respectively, and s is the equivalent stirrup stress (per unit area normal to the stirrups). The state of stress in the concrete corresponds to a uniaxial compression σ_b inclined at the angle φ to the x-axis (fig. 6).

We now make the additional assumption that the stress is homogeneous throughout the beam, i.e. σ_b and φ are constants. The three equations of equilibrium at the section are then:

$$0 = F - C - \sigma_b bh \cos^2 \varphi \qquad (4a)$$

$$V = \sigma_b bh \cos \varphi \sin \varphi \qquad (4b)$$

$$M = hF - \frac{1}{2} \sigma_b bh^2 \cos^2 \varphi \qquad (4c)$$

We assume that the conditions at the support are such that the boundary conditions for σ_x and τ_{xy} can be satisfied at x = 0. The conditions at the load (x = a) will be considered below. The boundary conditions at the lower and upper faces of the beam are:

$$\tau_{xy} = -\frac{1}{b}\frac{dF}{dx} \quad \text{and} \quad \sigma_y = 0 \quad \text{at} \quad y = 0$$

$$\tau_{xy} = -\frac{1}{b}\frac{dC}{dx} \quad \text{and} \quad \sigma_y = 0 \quad \text{at} \quad y = h$$

Using the relation V = dM/dx, we note from equations (4) that the boundary conditions for τ_{xy} are satisfied. The conditions for σ_y imply that the stress distribution is statically admissible, provided we put

$$s = \sigma_b \sin^2 \varphi \qquad (5)$$

Inserting into the equilibrium equations (4), we get:

$$0 = F - C - sbh\cot^2 \varphi \qquad (6a)$$

$$P = V = sbh\cot\varphi \qquad (6b)$$

$$Px = M = hF - \frac{1}{2} sbh^2 \cot^2 \varphi = h(F - \frac{1}{2} V\cot\varphi) \qquad (6c)$$

where we have expressed the stress resultants V and M by the external load P. Eliminating $\cot\varphi$ between equations (6b) and (6c) we obtain an equation for the load:

$$2Fbhs = P^2 + 2Pbxs$$

Since s is constant by equation (5), only F varies with x and we note that the maximum value $F = F_M$ occurs for $x = a$. Hence

$$P = \sqrt{(bas)^2 + 2F_M bhs} - bas$$

is a lower bound for the load. P being an increasing function of s and F_M, the best lower bound is obtained for $s = s_y$ and $F_M = F_y$. Thus the lower-bound solution for the load-carrying capacity is:

$$\frac{\tau}{\sigma_c} = \sqrt{\left(\frac{a}{h}\psi\right)^2 + 2\eta\psi} - \frac{a}{h}\psi \qquad (7)$$

Here we have introduced the parameters τ, ψ, and η defined as:

$$\tau = \frac{P}{bh} \quad ; \quad \psi = \frac{s_y}{\sigma_c} \quad ; \quad \eta = \frac{F_y}{bh\sigma_c}$$

Thus τ is the nominal shear stress while ψ and η are dimensionless parameters describing the strength of shear and longitudinal reinforcement, respectively.

Equation (7) expresses τ/σ_c as a monotonously increasing function of ψ, and we note that:

$$\frac{\tau}{\sigma_c} \to \frac{h}{a}\eta \qquad \text{for} \quad \frac{\psi}{\eta} \to \infty$$

Thus, as the shear reinforcement is increased, the load tends asymptotically to the yield load in pure flexion, $P_F = hF_y/a$. For small values of ψ the applicability of equation (7) may be restricted by bounds on $\cot\varphi$, depending on the conditions of support and loading.

The lower-bound solution of equation (7) is not very realistic, due to the fact that the critical section with yielding of the main reinforcement is at $x = a$ where the assumed homogeneous stress distribution is highly improbable. In order to derive a better lower bound we modify the stress distribution as indicated on fig. 7, assuming a stress-free region under the load. Thus at $x = a$ we have $C = F = F_M = Pa/h$.

According to equation (6c):

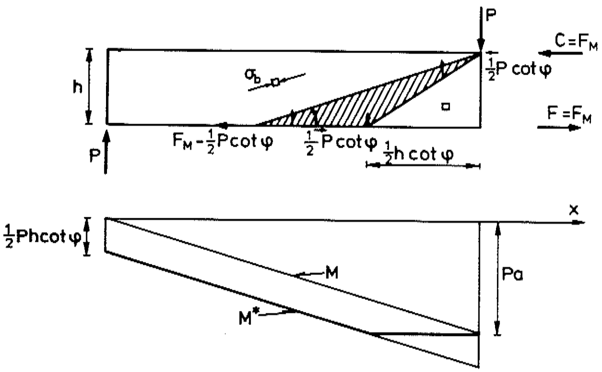


Fig. 7. Modified stress distribution and tensile stringer force for beam with concentrated loading.

$$F = \frac{Px}{h} + \frac{1}{2}\,P\cot\varphi$$

Hence $F = Pa/h$ at the section $x = a - 1/2\;h\cot\varphi$ and we assume that F remains constant for $a - 1/2\;h\cot\varphi \leqslant x \leqslant a$. The shaded region on fig. 7 (where the stress state is not homogeneous) is in equilibrium under the applied loads, provided we assume that it carries the vertical load P and the part $1/2\;P\cot\varphi$ of the compressive stringer force $C = F_M$ (cf. fig. 7). The state of stress in the shaded region is considered in detail by Nielsen [1].

Putting $F_M = F_y$, we now get $P = hF_y/a = P_F$, i.e. when the main reinforcement is yielding the beam has reached its flexural capacity. If the beam is to fail before that, it must be because the inclined concrete compressive stress becomes critical. Requiring $\sigma_b = \sigma_c$, we get from equation (5):

$$\cot\varphi = \sqrt{\frac{\sigma_c}{s} - 1}$$

The load P is then determined by equation (6b):

$$P = sbh\sqrt{\frac{\sigma_c}{s} - 1} = bh\sqrt{s(\sigma_c - s)}$$

This equation determines P as an increasing function of s as long as $s < \sigma_c/2$. Hence the best lower bound is obtained for $s = s_y$, i.e.:

$$\cot\varphi = \sqrt{\frac{1}{\psi} - 1} \tag{8}$$

and

$$\frac{\tau}{\sigma_c} = \sqrt{\psi(1-\psi)} \tag{9a}$$

for $\psi < 1/2$.

For $\psi \geqslant 1/2$ the optimum equivalent stirrup stress is $s = \sigma_c/2$ and we get:

$$\frac{\tau}{\sigma_c} = \frac{1}{2} \tag{9b}$$

Thus in this case the stirrups do not yield.

For small values of ψ, equation (9a) may be restricted by bounds on the value of $\cot\varphi$, depending on the support conditions.

Fig. 7 also shows the variation of the tensile stringer force which has to be carried by the longitudinal reinforcement. As is well known, it is not sufficient to design the main reinforcement according to the pure moment curve. Outside the vicinity of the maximum moment the tensile stringer force should be derived from the modified moment distribution $M^\star$ obtained from equation (6c):

$$hF = M^\star = M + \tfrac{1}{2}\,Vh\cot\varphi \tag{10}$$

where $\cot\varphi$ is given by equation (8).

The compressive stringer force is also given by equations (6):

$$C = F - V\cot\varphi = \frac{1}{h}M - \tfrac{1}{2}\,V\cot\varphi \tag{11}$$

We note that for $x < 1/2\ h\cot\varphi$ the compressive stringer force is negative, meaning that we get tension in the compressive flange, as is observed during tests. The absence of reinforcement to resist this tensile force does not lead to a decrease in the load-carrying capacity since we may modify the stress distribution at the support in a similar way as at the maximum moment (cf. Nielsen [1]).

3.2 Upper-Bound Solution

To find an upper bound for the load P we assume a failure mechanism consisting of a vertical shear deformation at an inclined yield line (fig. 8a). Denoting the inclination of the yield line by θ and the relative displacement rate by δ, we find the rate of internal work (plastic dissipation):

$$W_I = \delta s_y \, bh\cot\theta + \frac{\delta h}{\sin\theta} \cdot \frac{1}{2} b\sigma_c(1 - \cos\theta) \qquad (12)$$

Here the first term is the contribution from the stirrups crossing the yield line and the second term is obtained from equation (1). The rate of external work being

$$W_E = P\delta$$

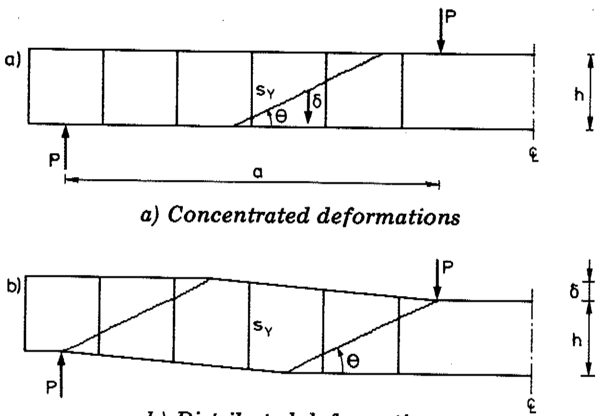
we find the upper bound to be:

$$P = s_y \, bh\cot\theta + \frac{1}{2}\sigma_c bh\left(\frac{1}{\sin\theta} - \cot\theta\right) = \frac{1}{2}\sigma_c bh\sqrt{1 + \cot^2\theta}$$

$$- \left(\frac{1}{2}\sigma_c - s_y\right) bh\cot\theta$$

On non-dimensional form:

$$\frac{\tau}{\sigma_c} = \frac{1}{2}\sqrt{1 + \cot^2\theta} - \left(\frac{1}{2} - \psi\right)\cot\theta = \frac{1}{2}\left(\sqrt{1 + \cot^2\theta} - \rho\cot\theta\right)$$



a) Concentrated deformations

b) Distributed deformations

Fig. 8. Failure mechanism for beam with concentrated loading.

where we have introduced the additional parameter

$$\rho = 1 - 2\psi$$

The optimal inclination of the yield line and the lowest upper bound is obtained from the condition:

$$\frac{dP}{d(\cot\theta)} = 0$$

which gives:

$$\cot\theta = \frac{\rho}{\sqrt{1-\rho^2}} = \frac{1-2\psi}{2\sqrt{\psi(1-\psi)}} \tag{13}$$

and

$$\frac{\tau}{\sigma_c} = \frac{1}{2}\sqrt{1-\rho^2} = \sqrt{\psi(1-\psi)} \tag{14a}$$

Equation (12) is valid as long as there is extension of the stirrups, i.e. $\cot\theta \geqslant 0$ or $\psi \leqslant 1/2$. For $\psi > 1/2$ the lowest upper bound is obtained for $\cot\theta = 0$, whence:

$$\frac{\tau}{\sigma_c} = \frac{1}{2} \qquad \text{for} \qquad \psi \geqslant \frac{1}{2} \tag{14b}$$

For small values of ψ the geometry of the beam poses the restriction $\cot\theta \leqslant a/h$, from which:

$$\frac{\tau}{\sigma_c} = \frac{\sqrt{a^2+h^2} - (1-2\psi)a}{2h} \tag{14c}$$

for

$$\psi \leqslant \frac{\sqrt{a^2+h^2} - a}{2\sqrt{a^2+h^2}} = \psi_0$$

The value ψ_0 is rather small. For a/h = 3 we get:

$$\psi_0 = \frac{\sqrt{a^2+h^2} - a}{2\sqrt{a^2+h^2}} = \frac{\sqrt{10}-3}{2\sqrt{10}} \cong 0.0256$$

Taking the ratio between stirrup yield stress and concrete cylinder strength to be 15, this corresponds to a shear reinforcement ratio of less than 0.2%.

The upper bound, equations (14), and the lower bound, equations (9), are plotted on fig. 9a in the case of a/h = 3. We note that for $\psi \geqslant \psi_0$ the upper-bound solution coincides with the lower-bound

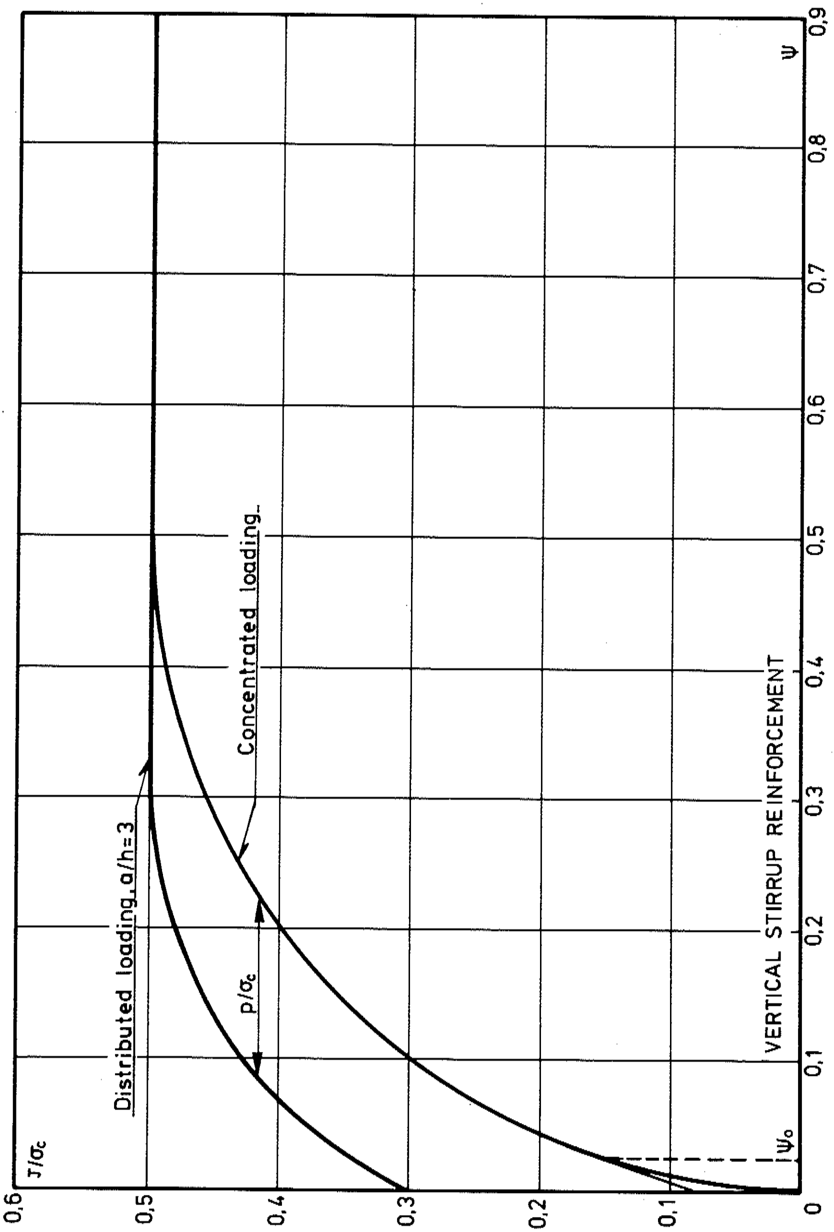


Fig. 9a. Web-crushing criterion for beams with vertical stirrups.

solution, equations (9), found above. This means that the web crushing criterion gives the correct plastic load-carrying capacity according to the assumptions made.

For $\psi < \psi_0$ the web crushing criterion is only an upper bound, still it gives a fairly good description of the shear strength, even for beams without any shear reinforcement. Putting $\psi = 0$, we find:

$$\frac{\tau}{\sigma_c} = \frac{\sqrt{a^2 + h^2} - a}{2h}$$

In accordance with experimental evidence★, the shear strength is a decreasing function of the shear span ratio a/h. The curve is plotted on fig. 9b.

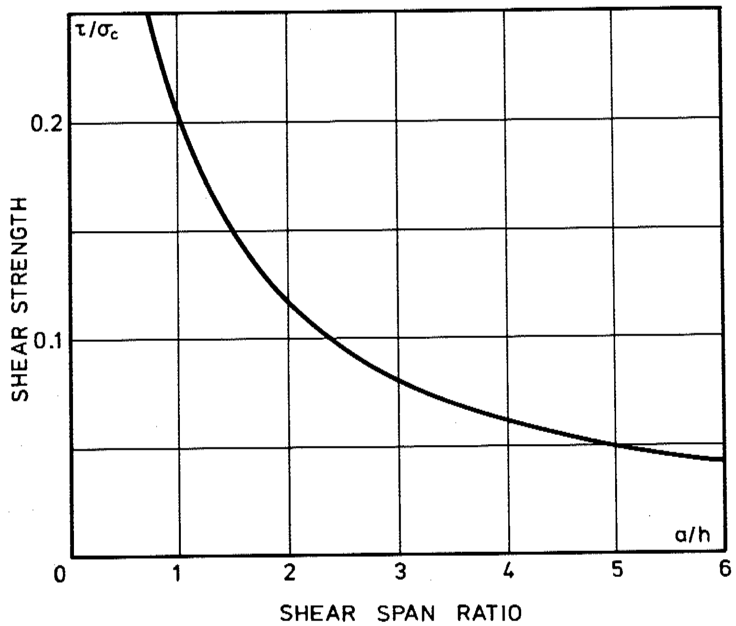


Fig. 9b. Web-crushing criterion for beams without shear reinforcement.

★ As for shear-reinforced beams an effective concrete strength $\sigma_c^\star = \nu\sigma_c$, see section 6, has to be introduced. Test results on beams with rectangular section indicate $\nu \approx 0.6$, when h is put equal to the full height of the section. Beams without shear reinforcement will not be further treated here.

Comparing figs. 5, 6, and 8a we identify the angles β and γ of fig. 5 as:

$$\beta = \theta \quad \text{and} \quad \gamma = \theta - \varphi$$

Hence equation (2) yields:

$$\theta = 2(\theta - \varphi) \quad \text{or} \quad \theta = 2\varphi$$

We note from equations (8) and (13) that this is indeed the case.

For mathematical convenience we have assumed the deformation to be concentrated in a diagonal yield line. As seen on fig. 8a, the yield line may be at an arbitrary position between the load and the support. We might as well assume the deformations to be distributed over the region between load and support, as shown on fig. 8b.

4. *DISTRIBUTED LOADING*

4.1 *Lower-Bound Solution*

We now turn to the case of a simply supported beam loaded symmetrically by an equally distributed load p per unit area of the shear span (fig. 10).

To derive a lower-bound solution we assume a state of stress in the beam as given by equations (3) with φ constant. However, since the boundary conditions for σ_y now require:

$$\sigma_b \sin^2\varphi = s \qquad \text{at} \quad y = 0$$

and

$$\sigma_b \sin^2\varphi = s + p \qquad \text{at} \quad y = h$$

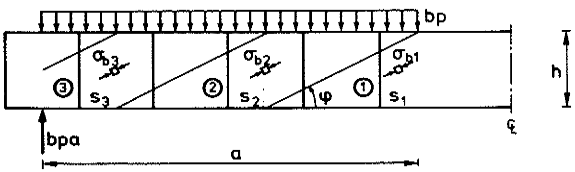


Fig. 10. Stress distribution for simple beam with distributed loading.

we can no longer take σ_b and s to be constants. We construct a statically admissible stress distribution, proceeding as indicated by Nielsen [1], assuming σ_b constant in n regions bounded by the beam faces and planes inclined at the angle φ and vertically spaced at the distance h, starting at the section x = a (fig. 10). Similarly, s is assumed constant in the corresponding n—1 intervals along the x-axis. Introducing the notation σ_{bk} for the compressive concrete stress in the k^{th} region and s_k for the equivalent stirrup stress in the k^{th} interval (cf. fig. 10), we find from the boundary conditions for σ_y:

$$\sigma_{bk} \sin^2 \varphi = s_k = s_{k-1} + p \tag{15}$$

The boundary conditions for τ_{xy} require:

$$\text{at } y = 0: \quad \frac{dF_k}{dx} = - b\tau_{xyk} = b\sigma_{bk} \cos\varphi \sin\varphi = bs_k \cot\varphi$$

$$\text{at } y = h: \quad \frac{dC_k}{dx} = - b\tau_{xyk+1} = b\sigma_{bk+1} \cos\varphi \sin\varphi = b(s_k + p) \cot\varphi$$

We note that while σ_b and s decrease with increasing x, then F and C are increasing functions of x, varying linearly within each interval k. Hence the maximum value F_M of the tensile stringer force is obtained at the section x = a in the first interval. On the other hand σ_{b1} and s_1 are the minimum values of the compressive concrete stress and the equivalent stirrup stress, respectively.

We assume $\sigma_{b1} = 0$ and $s_1 = 0$. Hence in the k^{th} interval, i.e. for $a - x = \lambda h \cot\varphi$, where $k - 1 \leqslant \lambda < k$, we find after some manipulation:

$$\sigma_{bk} \sin^2 \varphi = s_k = (k - 1)p$$

$$F_k = F_M - (k - 1)\left(\lambda - \frac{k}{2}\right) bph \cot^2 \varphi$$

The equilibrium equations at an arbitrary section of the k^{th} interval can now be established (cf. equations (6)):

$$0 = F_k - C_k - (k - \lambda)\sigma_{bk} bh\cos^2 \varphi - (\lambda - k + 1)\sigma_{bk+1} bh\cos^2 \varphi$$

$$= F_M - C_k - k\left[\lambda - \frac{1}{2}(k - 1)\right] bph^2 \cot^2 \varphi \tag{16a}$$

$$bp(a-x)=V=(k-\lambda)\sigma_{bk}h\cos\varphi\sin\varphi+(\lambda-k+1)\sigma_{bk+1}bh\cos\varphi\sin\varphi$$

$$=[s_k+(\lambda-k+1)p]bh\cot\varphi=\lambda pbh\cot\varphi \tag{16b}$$

$$\frac{1}{2}bpx(2a-x)=M=hF_k-(\frac{1}{2}\lambda-\frac{1}{2}k+1)(k-\lambda)\sigma_{bk}bh^2\cos^2\varphi$$

$$-\frac{1}{2}(\lambda-k+1)^2\sigma_{bk+1}bh^2\cos^2\varphi$$

$$=hF_k-\frac{1}{2}[s_k+(\lambda-k+1)^2p]bh^2\cot^2\varphi$$

$$=hF_M-\frac{1}{2}\lambda^2bph^2\cot^2\varphi \tag{16c}$$

Equation (16b) is identically satisfied whereas equation (16c) determines the load. Considering the section $x = a$ $(k = 1, \lambda = 0)$, we get:

$$\frac{1}{2}bpa^2 = M = hF_M$$

The best lower bound is obtained for $F_M = F_y$, i.e.

$$p=\frac{2hF_y}{ba^2}=p_F$$

We conclude that if the beam fails by yielding of the main reinforcement, it has reached the flexural capacity p_F. If the beam fails in shear it must be because the maximum compressive concrete stress has reached the cylinder strength, i.e. $\sigma_n = \sigma_c$. Hence, by equation (15)

$$\sigma_c\sin^2\varphi = s_M + p$$

i.e.

$$\cot\varphi=\sqrt{\frac{\sigma_c}{s_M+p}-1} \tag{17}$$

where $s_M = s_{n-1}$ is the maximum equivalent stirrup stress (in the last interval). The shear force at the support $x = 0$ is found from equation (16b):

$$bpa = (s_M + \kappa p)bh\sqrt{\frac{\sigma_c}{s_M + p} - 1}$$

where $\kappa = \dfrac{a}{h\cot\varphi} - n + 2$ is a factor $(0 < \kappa \leqslant 1)$ which accounts for the fact that the shear span a is not necessarily a multiplum of $h\cot\varphi$. Assuming $\kappa = 1$, we get:

$$pa = h\sqrt{(s_M + p)(\sigma_c - s_M - p)}$$

with the solution

$$p = \frac{h^2}{2(a^2 + h^2)}\left[\sigma_c - 2s_M + \sqrt{\sigma_c^2 + 4\frac{a^2}{h^2}s_M(\sigma_c - s_M)}\right]$$

The load p is an increasing function of s_M as long as $s_M \geqslant 1/2\ \sigma_c(1 - h/a)$. Hence for $0 < \psi < 1/2(1 - h/a)$ the best lower bound is obtained for $s_M = s_y$, i.e.:

$$\frac{\tau}{\sigma_c} = \frac{ah}{2(a^2 + h^2)}\left[1 - 2\psi + \sqrt{1 + 4\frac{a^2}{h^2}\psi(1 - \psi)}\right] \qquad (18a)$$

Here we have introduced the nominal shear $\tau = pa/h$. For $\psi \geqslant 1/2(1 - h/a)$ the stirrups do not yield, and we get

$$\frac{\tau}{\sigma_c} = \frac{1}{2} \qquad (18b)$$

In general we will have $\kappa < 1$ corresponding to the situation shown on fig. 10 where the n^{th} stress region does not reach the tensile stringer. Thus eqs. (18) are not a true lower bound, since the approximation $\kappa = 1$ is not a conservative one. However, as the compressive concrete stress is concentrated above the main reinforcing bars (except for I-beams) we would not expect web crushing to be critical before the cylinder strength is reached at the level of the main reinforcement, in which case our assumption is justified.

The tensile stringer force is shown on fig. 11. We note from equations (16) that only at the interval points do we have $hF = M^\star$, where $M^\star$ is the modified moment distribution defined by equation (10), and $\cot\varphi$ is given by equation (17). This is due to the fact that we require F to vary linearly within each interval.

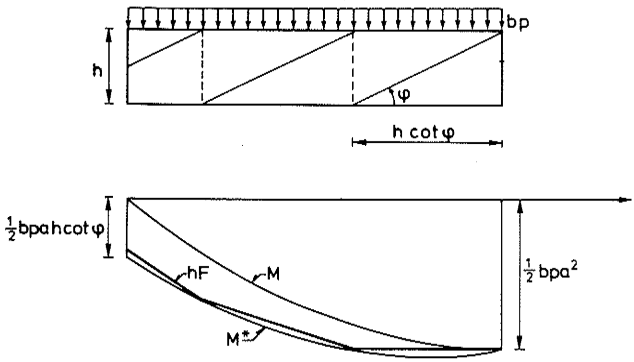


Fig. 11. Tensile stringer force for beam with distributed loading.

Also equation (11) for the compressive stringer force holds at the interval points only. As with concentrated loading, we get a region near the support with tension in the compression flange.

4.2. Upper-Bound Solution

We derive an upper bound for the yield load assuming a failure mechanism consisting of an inclined yield line with the inclination θ starting at the support (fig. 12). The vertical displacement rate being δ, the rate of internal work is given by equation (12):

$$W_I = \delta s_y bh\cot\theta + \frac{1}{2}\delta\sigma_c bh\frac{1-\cos\theta}{\sin\theta}$$

The rate of external work is:

$$W_E = \delta bp(a - h\cot\theta)$$

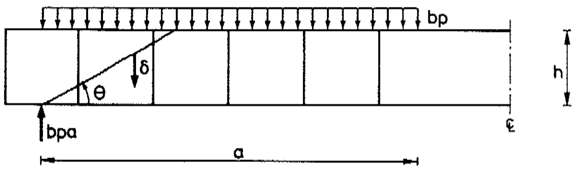


Fig. 12. Failure mechanism for beam with distributed loading.

Equating the two expressions, we find an upper bound for the load p:

$$p = \frac{hs_y \cot\theta + \frac{1}{2} h\sigma_c (\frac{1}{\sin\theta} - \cot\theta)}{a - h\cot\theta}$$

On non-dimensional form:

$$\frac{\tau}{\sigma_c} = \frac{\frac{1}{2}(\sqrt{1+\cot^2\theta} - \rho\cot\theta)a}{a - h\cot\theta}$$

The minimum upper bound is found requiring $d\tau/d\cot\theta = 0$, leading to:

$$\cot\theta = \frac{1}{a(1-\rho^2)}[\rho\sqrt{a^2(1-\rho^2)+h^2} - h] \qquad (19)$$

and

$$\frac{\tau}{\sigma_c} = \frac{ah}{2(a^2+h^2)}[1 - 2\psi + \sqrt{1 + 4\frac{a^2}{h^2}\psi(1-\psi)}] \qquad (20a)$$

We note that $\cot\theta \geqslant 0$ for $h/a \leqslant \rho \leqslant 1$. Hence for large values of ψ, equation (20a) is replaced by:

$$\frac{\tau}{\sigma_c} = \frac{1}{2} \quad \text{for} \quad \psi \geqslant \frac{1}{2}(1 - \frac{h}{a}) \qquad (20b)$$

there is no restriction for small values of ψ since

$$\rho \to 1 \Rightarrow \cot\theta \to \frac{a^2 - h^2}{2ah} < \frac{a}{h} \quad (\text{in fact } \cot(\frac{\theta}{2}) = \frac{a}{h})$$

and

$$\frac{\tau}{\sigma_c} = \frac{ah}{a^2+h^2} \quad \text{for} \quad \psi = 0$$

We might expect to obtain a smaller upper bound, considering a curved yield line. Using calculus of variations it is easy to show, however, that of all curves starting at the support, the straight line is in fact optimal. Indeed, we note that the upper bound, equations (20), is identical to the lower bound, equations (18). Hence we have determined the correct yield load. Accordingly, equation (17) with $s_M = s_y$ yields:

$$\cot^2\varphi = \frac{1}{\psi + \frac{h\tau}{a\sigma_c}} - 1 = \frac{a^2(1+\rho) + h^2 - h\sqrt{a^2(1-\rho^2) + h^2}}{a^2(1-\rho) + h^2 + h\sqrt{a^2(1-\rho^2) + h^2}}$$

This may be reduced to:

$$\cot\varphi = \frac{a(1+\rho)}{h + \sqrt{a^2(1-\rho^2) + h^2}}$$

On the other hand, by equation (19):

$$\tan\frac{\theta}{2} = \sqrt{1 + \cot^2\theta} - \cot\theta$$

$$= \frac{1}{a(1-\rho)^2}\left[\sqrt{a^2(1-\rho^2) + h^2} - h\rho - \rho\sqrt{a^2(1-\rho^2) + h^2} + h\right]$$

$$= \frac{h + \sqrt{a^2(1-\rho^2) + h^2}}{a(1+\rho)}$$

Thus we have $\theta = 2\varphi$, as we would expect.

The yield load as a function of the shear reinforcement, equations (20), is plotted on fig. 9a for the case of a/h = 3.

Inversion of equation (20a) yields:

$$\psi = \frac{1}{2}\left[1 - \frac{2h}{a}\frac{\tau}{\sigma_c} - \sqrt{1 - 4(\tau/\sigma_c)^2}\right]$$

whereas inversion of equation (14a) gives:

$$\psi = \frac{1}{2}\left[1 - \sqrt{1 - 4(\tau/\sigma_c)^2}\right]$$

Hence the distance between the two curves measured parallel to the ψ-axis equals $h\tau/a\sigma_c = p/\sigma_c$ (cf. fig. 9a). This means that in order to get the same shear capacity by distributed loading as by concentrated loading we may reduce the necessary shear reinforcement s_y by the amount p. We also note, comparing equations (19) and (13), that the yield line is steeper by distributed loading.

5. INCLINED STIRRUPS

In the preceding sections we have assumed the stirrups to be vertical. We now modify the model, considering stirrups inclined at an angle α $(0 < \alpha \leqslant \frac{\pi}{2})$ to the beam axis (fig. 13). We still assume that the action of the stirrups can be described by an equivalent stirrup stress s, per unit area normal to the stirrups. The yield stress s_y is then proportional to the volume of shear reinforcement, if we neglect the amount of steel required for anchoring the stirrups (including the horizontal legs in the case of closed stirrups).

For simplicity we consider only the case of concentrated loading. The stress distribution, equations (3), is modified to:

$$N_x = -C \qquad \text{for } y = h$$

$$\left.\begin{aligned} \sigma_x &= -\sigma_b \cos^2\varphi + s\cos^2\alpha \\ \sigma_y &= -\sigma_b \sin^2\varphi + s\sin^2\alpha \\ \tau_{xy} &= -\sigma_b \cos\varphi \sin\varphi - s\cos\alpha \sin\alpha \end{aligned}\right\} \qquad \text{for } 0 < y < h$$

$$N_x = F \qquad \text{for } y = 0$$

The equilibrium equations (4) now become:

$$0 = F - C - \sigma_b bh\cos^2\varphi + sbh\cos^2\alpha$$

$$V = \sigma_b bh\cos\varphi\sin\varphi + sbh\cos\alpha\sin\alpha$$

$$M = hF - \frac{1}{2}\sigma_b bh^2\cos^2\varphi + \frac{1}{2} sbh^2\cos^2\alpha$$

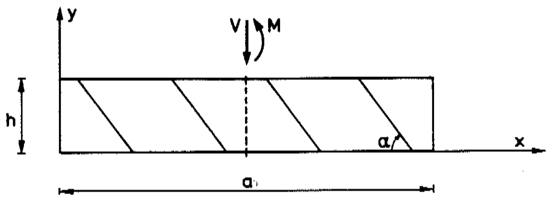


Fig. 13. Beam with inclined stirrups.

We note that the boundary conditions for τ_{xy} at y = 0 and y = h are satisfied. The boundary conditions for σ_y imply:

$$s \sin^2\alpha = \sigma_b \sin^2\varphi$$

The counterparts of equations (6) are then:

$$0 = F - C - sbh\sin^2\alpha(\cot^2\varphi + \cot^2\alpha)$$

$$P = V = sbh\sin^2\alpha(\cot\varphi + \cot\alpha)$$

$$Px = M = hF - \frac{1}{2} sbh^2 \sin^2\alpha(\cot^2\varphi - \cot^2\alpha)$$
$$= h[F - \frac{1}{2} V(\cot\varphi - \cot\alpha)]$$

As before we modify the stress distribution in the vicinity of the load, requiring $F = F_M = Pa/h$ in the interval $a - \frac{1}{2} h(\cot\varphi - \cot\alpha) < x < a$. Thus shear failure occurs by crushing of the concrete, whence:

$$\cot\varphi = \sqrt{\frac{\sigma_c}{s \sin^2\alpha} - 1}$$

and the lower bound is:

$$\tau = \sqrt{s \sin^2\alpha(\sigma_c - s \sin^2\alpha)} + s \cos\alpha\sin\alpha$$

This is an increasing function of s as long as $s \leqslant \frac{1}{2}\sigma_c(1 + \cos\alpha)/\sin^2\alpha$. Hence the maximum lower bound is obtained for $s = s_y$, i.e.

$$\cot\varphi = \sqrt{\frac{1}{\psi \sin^2\alpha} - 1} \tag{21}$$

and

$$\frac{\tau}{\sigma_c} = \sqrt{\psi \sin^2\alpha(1 - \psi \sin^2\alpha)} + \psi \cos\alpha\sin\alpha \tag{22a}$$

for $\psi \leqslant \frac{1}{2}(1 + \cos\alpha)\sin^2\alpha$.

For $\psi > \frac{1}{2}(1 + \cos\alpha)/\sin^2\alpha$ the maximum lower bound is obtained for $s = \frac{1}{2}\sigma_c(1 + \cos\alpha)/\sin^2\alpha$, i.e. the stirrups are not yielding at failure of the beam, and we get

$$\frac{\tau}{\sigma_c} = \frac{1 + \cos\alpha}{2\sin\alpha} = \frac{1}{2}\cot\frac{\alpha}{2} \tag{22b}$$

To find an upper bound we note that equation (12) for the rate of internal work in a failure mechanism consisting of a vertical displacement rate δ in an inclined yield line with the inclination θ is modified to:

$$W_I = \delta s_y \sin^2\alpha bh(\cot\theta + \cot\alpha) + \frac{\delta h}{\sin\theta}\frac{1}{2}b\sigma_c(1 - \cos\theta)$$

Hence the upper bound for the load becomes:

$$\frac{\tau}{\sigma_c} = \psi \sin^2\alpha(\cot\theta + \cot\alpha) + \frac{1}{2}(\sqrt{1 + \cot^2\theta} - \cot\theta)$$

The condition $d\tau/d(\cot\theta) = 0$ yields the inclination:

$$\cot\theta = \frac{1 - 2\psi\sin^2\alpha}{2\sqrt{\psi\sin^2\alpha(1 - \psi\sin^2\alpha)}} \qquad (23)$$

This leads to the minimum upper bound:

$$\frac{\tau}{\sigma_c} = \sqrt{\psi\sin^2\alpha(1 - \psi\sin^2\alpha)} + \psi\cos\alpha\sin\alpha$$

Thus the upper bound equals the lower bound, equation (22a), and comparing equations (21) and (23) we note that $\theta = 2\varphi$, as we would expect.

The requirement that the stirrups be in tension, i.e. $\cot\theta + \cot\alpha \geq 0$, now gives $\psi \leq \frac{1}{2}(1 + \cos\alpha)/\sin^2\alpha$. For ψ greater than this value, the lowest upper bound is obtained for $\cot\theta = -\cot\alpha$, i.e.

$$\frac{\tau}{\sigma_c} = \frac{1}{2}(\sqrt{1 + \cot^2\alpha} + \cot\alpha) = \frac{1}{2}\cot\frac{\alpha}{2}$$

as found above, equation (22b).

The upper bound is valid as long as $\cot\theta \leq a/h$. For $\cot\theta = a/h$ we find:

$$\frac{\tau}{\sigma_c} = \frac{\sqrt{a^2 + h^2} - (1 - 2\psi\sin^2\alpha)a}{2h} + \psi\cos\alpha\sin\alpha \qquad (22c)$$

which holds for $\psi < \psi_0$, where ψ_0 is now given as:

$$\psi_0 = \frac{1}{\sin^2\alpha}\frac{\sqrt{a^2 + h^2} - a}{2\sqrt{a^2 + h^2}}$$

With respect to failure mechanism the effect of stirrup inclination amounts to a replacement of the parameter ψ in equations (8) and (13) by $\psi \sin^2\alpha$. For $\alpha < \pi/2$ this leads to increased values of $\cot\varphi$ and $\cot\theta$, i.e. the yield line is flatter by inclined than by vertical stirrups. Concerning the load-carrying capacity we see by comparing equations (9) and (22) that the inclined stirrups are most effective for

$$\psi > \frac{\cos^2\alpha}{1 + 3\sin^2\alpha}$$

assuming $\psi \geqslant \psi_0$.
In the important case $\alpha = \pi/4$, the inclined stirrups are more effective than vertical stirrups for $\psi > 0.2$.

The optimal stirrup inclination $\alpha = \alpha_M$ is found from the condition $d\tau/d\alpha = 0$, where τ is given by equations (22).

From equation (22a) we get

$$\cot\alpha_M = \sqrt{\psi}$$

Hence the maximum shear strength $\tau = \tau_M$ that can be obtained by a given amount ψ of shear reinforcement is given by:

$$\tau_M/\sigma_c = \sqrt{\psi}$$

Since

$$\frac{1 + \cos\alpha_M}{2\sin^2\alpha_M} = \frac{1}{2}\sqrt{1 + \psi}\,(\sqrt{1 + \psi} + \sqrt{\psi}) > \psi$$

this holds for all high values of ψ. Thus for thin-webbed beams, a substantial increase of shear strength can be obtained by placing the stirrups at a suitable angle. This is in accordance with experience.

For small values of ψ, we find from equation (22c)

$$\cot\alpha_M = \frac{\sqrt{a^2 + h^2} - a}{h}$$

and

$$\frac{\tau_M}{\sigma_c} = \frac{\sqrt{a^2 + h^2} - a + (\sqrt{a^2 + h^2} + a)\psi}{2h}$$

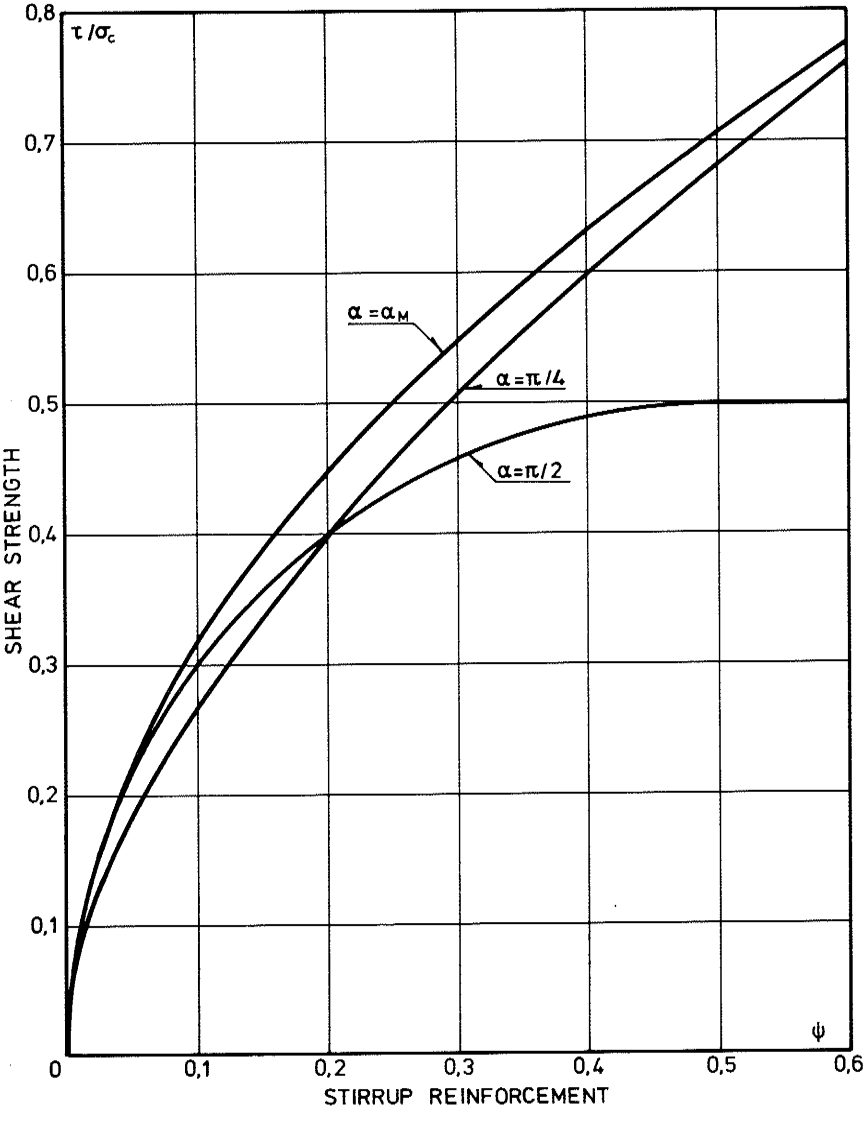


Fig. 14. Web-crushing criterion for beam with inclined stirrups.

valid for

$$\psi < \psi_0 = \left(\frac{\sqrt{a^2 + h^2} - a}{h}\right)^2$$

Since $\varphi = \theta/2$, we note that in both cases:

$$\alpha_M = \frac{\pi}{2} - \varphi$$

This means that when the stirrup inclination is optimal, then the direction of the uniaxial concrete compression is normal to the stirrups.

On fig. 14 the web-crushing criterion is plotted for stirrup inclinations $\alpha = \pi/2$, $\alpha = \pi/4$, and $\alpha = \alpha_M$.

Equation (22c) corresponding to $\psi < \psi_0$ has been omitted on the graph.

6. *TEST RESULTS*

In the following we shall consider beams with vertical stirrups and with fairly strong shear reinforcement, i.e. $\psi \geqslant \psi_0$. The web-crushing criterion, equations (9), yields:

$$V = h\sqrt{bs_y(b\sigma_c - bs_y)} \quad \text{for} \quad bs_y \leqslant \frac{1}{2} b\sigma_c$$

$$V = \frac{1}{2} h\, b\sigma_c \quad \text{for} \quad bs_y > \frac{1}{2} b\sigma_c$$

Thus, in order to determine the shear strength, we need know three parameters:

- h: The shear depth, i.e. the distance between the tension and the compression stringer.
- $b\sigma_c$: The compressive strength of the web, per unit length of the beam.
- bs_y: The yield strength of the shear reinforcement, per unit length of the beam.

Of these quantities, only the last is well defined, and may be found by tension tests on the stirrup steel. The uniaxial, compressive concrete strength σ_c may be determined by cylinder tests, but we cannot except the quantity $b\sigma_c$ to be an adequate measure of the web strength. This is due to the fact that the compression is applied to the web concrete through the longitudinal bars. This concentration of the load leads to failure of the concrete at a stress level which, as an average over the web, is less than σ_c. Thus we are led to the introduction of the effective web strength $b\sigma_c^\star = \nu b\sigma_c$, where ν is a web effectiveness parameter.

The definition of h is not too precise either, since at least the compression stringer has a considerable extension. Therefore, we introduce the effective shear depth $h^\star$. Most European and American standards relate the allowable shear stress to the effective depth of beam. The Danish Code of Practice DS 411 [13], [14] requires the use of the internal moment arm z, calculated at the section of maximum moment. Another possibility would be to choose as effective shear depth the depth of the web, i.e. of the concrete body assumed to carry the shear. In this case any contribution from the compression flange is neglected.

We define the effective shear strength

$$\tau^\star = \frac{V}{bh^\star}$$

where V is the applied shear force, b is the web width, and $h^\star$ is the effective shear depth. Introducing the web effectiveness parameter ν, the web-crushing criterion for beams with vertical stirrups is modified to:

$$\frac{\tau^\star}{\sigma_c} = \sqrt{\psi(\nu - \psi)} \qquad \text{for} \qquad \psi \leqslant \frac{\nu}{2} \tag{24a}$$

and

$$\frac{\tau^\star}{\sigma_c} = \frac{\nu}{2} \qquad \text{for} \qquad \psi > \frac{\nu}{2} \tag{24b}$$

Here ψ is the non-dimensional shear reinforcement strength $\psi = s_y/\sigma_c$, s_y being the yield strength of the shear reinforcement per unit area of a horizontal web section. In the presence of a distributed load p, per unit area of the web section, the parameter ψ should be replaced by $\psi + p/\sigma_c$.

Equation (24a) represents a circle with radius $\nu/2$ and centre at $(\psi, \tau^{\star}/\sigma_c) = (\nu/2, 0)$.

In order to assess the values of ν and $h^{\star}$, we compare equations (24) with experimental evidence. Note that the two parameters are interrelated in the sense that the choice of $h^{\star}$ will effect the value of ν and vice versa.

A good starting point would be to look for beams where actual web failure is observed, i.e. thin-webbed beams. Such beams have been tested by Robinson [11], Leonhardt & Walther [12], and Placas as reported by Regan [10]. Assuming the web to be fully effective ($\nu = 1$) the results indicate that the effective shear depth is the lesser value, i.e. the depth of the web. On the other hand, we do not know if the web actually sustains a load corresponding to the one-dimensional concrete strength, measured on cylinder or cube specimens. The fact that stirrups with high tension are embedded in the web might lead to a reduction in strength. Investigations by Robinson & Demorieux [15] and Demorieux [17], [18] do indeed show such an effect. Since experimental documentation is lacking, we shall adhere to common Danish practice, taking the effective shear depth equal to the internal moment lever arm. In the analysis of test results described below, the shear depth is measured from the centroid of the tensile reinforcement to the centre of the compressive flange.

Having decided upon $h^{\star}$, we can proceed to determine ν. To that end we need test results covering a wide range of shear reinforcement strengths, from the weak to the very strong. Unfortunately, most of the shear tests reported in the literature involve beams with little or no shear reinforcement.

A representative experimental investigation has been carried out by Leonhardt & Walther [16]. The results are plotted on fig. 15. The authors measured the cube strength β_w of the concrete, and we have assumed the cylinder strength σ_c to be $\sigma_c = 0.8\ \beta_w$. The test series comprised 18 beams of which three (beams TA5, TA17, and TA18) had bent-up bars as shear reinforcement in addition to vertical stirrups. Two beams (TA7 and TA8) had curtailed main reinforcement. Of the remaining 13 beams, three (beams TA9, TA10, and TA13) are omitted from the plot because they were reported to have failed in bending.

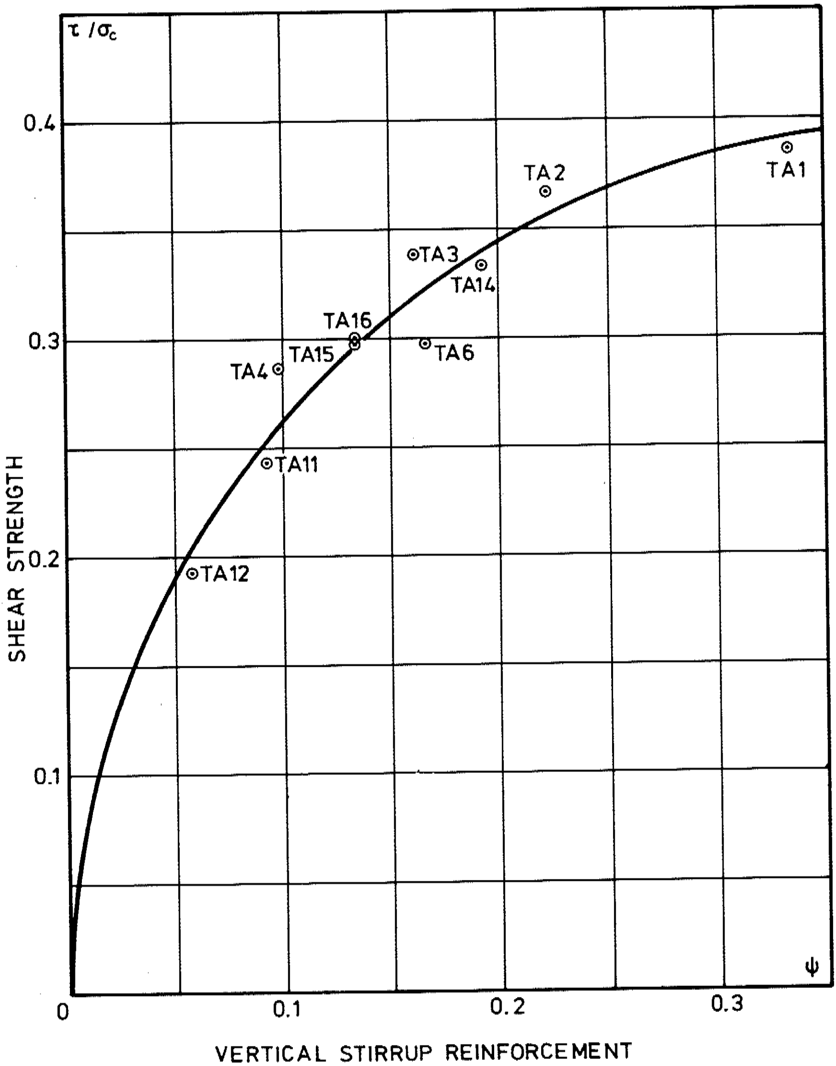


Fig. 15. Shear tests by Leonhardt & Walther [16].

The prediction of the theory that the points lie on a Mohr's circle through the origin, is fairly well supported by the experimental evidence. The circle giving closest fit (shown on fig. 15) corresponds to a web effectiveness $\nu = 0.796$, the coefficient of variation being 2.8%.

One of the beams (TA6) was provided with additional horizontal web reinforcement. Otherwise, the beam was identical with beam TA3, As expected, the horizontal reinforcement has no positive effect on the shear strength.

To demonstrate the general applicability of the theory, fig. 16 shows the results of 153 shear failures, including 66 tests carried out recently at the Structural Research Laboratory. The plot refers to simply supported T-beams (without tension flange). Only slender beams ($a/d > 2.5$, d being the effective depth) and beams with some shear reinforcement ($\psi > 0.01$) are considered and flexural failures are omitted. The graph includes a single test (from Leonhardt & Walther [12]) with distributed loading. The result has been plotted substituting $\psi + p/\sigma_c$ for ψ★.

The points are seen to be fairly well distributed about the web-crushing criterion corresponding to $\nu = 0.72$, although the scatter is considerable (the coefficient of variation is 6.0%). Fig. 16 also shows the shear strength calculated on basis of the Danish Code of Practice DS 411 [13], [14]. No safety factors have been introduced and average rather than characteristic strength parameters are used. Still, the code is seen to be very conservative, especially for small values of ψ.

In order to get a better prediction of the shear strength by the web-crushing criterion we must determine ν for each individual beam type. We would expect the web effectiveness to depend on the web width and on the reinforcement details, i.e. the number and dimensions of the longitudinal bars, their distances from the edges of the beam, and whether or not they are supported by stirrups. Currently, extensive test series are being carried out at the Structural Research Laboratory with the aim to study this question. The experiments should also be able to verify or falsify the following assumptions or consequences of the proposed theory:

★ Detailed documentation on fig. 16 is available from the authors, who would appreciate information about test series not included in the plot.

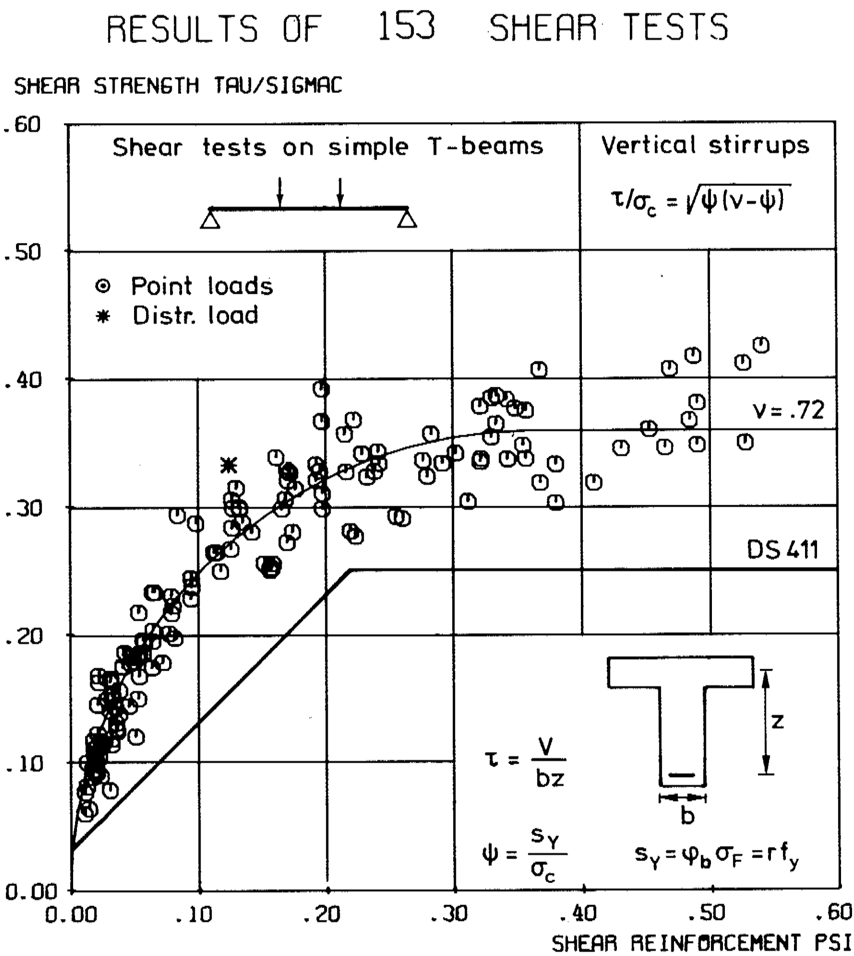


Fig. 16. Test results versus web-crushing criterion and code of practice.

a) For sufficiently strong shear reinforcement the shear resistance reaches a level depending only on concrete strength.

b) The shear strength is independent of the strength of the main reinforcement (though it does depend on the number and dimensions of reinforcing bars).

c) At yield the deformations consist of a vertical translation of the load section relative to the support section.

In order to study item (c) simultaneous measurements are taken of the compression of the flange, the elongation of the main reinforcement, and the deflections. This is done at constant load when the beam is near failure. The preliminary results indicate that assumption (c) is not far off, whereas the concept of a rotation about the crack head is totally misleading.

7. CONCLUSION

In the preceding sections we have established a rational theory for the shear strength of reinforced concrete beams with a compression flange. Simple beams with concentrated and distributed loading are considered. In the former case the stirrups may be vertical or inclined.

The derivation is based on the mathematical theory of plasticity and identical upper and lower bounds are found in the three cases mentioned above, equations (9), (18), and (22). As failure criterion for the concrete the square yield locus is adopted (fig. 4). The failure mechanism consists of a vertical displacement located at diagonal yield lines (fig. 8). The behaviour of the beam, as reflected by the theory, is as follows:

I: The first cracks follow the trajectories of principal stress in the uncracked beam. As the shear reinforcement takes over, the cracks tend to become flatter.

II: The stirrups are yielding. The deformations are located in yield lines composed of cracks and crushing zones. An increase in load is obtained by a flattening of the yield lines, causing more shear reinforcement to yield.

III: This process goes on until the compressive stress in the web becomes so high that the concrete fails.

IV: This may lead to actual web failure or to so great deformations that the compression flange is destroyed.

We note that the load-carrying capacity of the beams is reached in stage III, when the web concrete fails either by crushing or by spalling. Therefore, we have chosen the label web-crushing criterion for the load formula, though actual web distress need not be observed, the failure being a local phenomenon at the main reinforcing bars. The failure of the compression flange (shear compression failure) often observed, is regarded as a secondary phenomenon caused by excessive deformations after the failure load is reached.

The principal features of the proposed theory, by which it differs from other shear theories, are as follows:

A: The failure mechanism is not a rotation about the crack head but rather a vertical translation.

B: The theory gives a rational explanation for the shear stresses in the yield line (interface shear transfer) that contribute substantially to the load-carrying capacity.

Both the relevance of the failure mechanism and the existence of interface shear transfer is amply supported by test experience. It must be stressed, however, that it is a necessary condition for the web-crushing criterion to apply, that the main reinforcement does not yield at failure of the beam. For continuous beams with high shear and bending located at the same section, this may be different. Also, it should be noted that for weak shear reinforcement, the inclination of the web compression becomes very flat. This may lead to problems with anchorage and curtailment of the longitudinal reinforcing bars. In the laboratory, extraordinary measures may be taken to prevant bond failure, but this might very well prove too costly for engineering practice. It may therefore be advantageous to design the shear reinforcement for a steeper web compression than the one required by the web-crushing criterion.

ACKNOWLEDGEMENTS

Funds for the investigation presented in this paper are provided by the Danish Government Fund for Scientific and Industrial Research. Also, the authors wish to thank lektor A. Rathkjen of the Engineering Academy in Aalborg for many valuable suggestions and a thorough checking of the manuscript.

NOTATIONS

All symbols are defined when they first occur in the text. Definitions of the repeatedly used notations are listed below.

a: shear span

b: beam width

C: compressive stringer force

F: tensile stringer force

F_M: maximum tensile stringer force

F_y: tensile stringer yield force

h: beam depth

$h^\star$: effective shear depth

M: bending moment

$M^\star$: modified moment distribution for design of main reinforcement

N_x: stringer force

P: concentrated load

P_F: concentrated yield load in pure flexion hF_y/a

p: distributed load per unit area

p_F: distributed flexural failure load $2hF_y/a$

s: equivalent stirrup stress (per unit area normal to the stirrups)

s_M: maximum equivalent stirrup stress

s_y: equivalent stirrup yield stress

V: shear force

W_E: rate of external work
W_I: rate of internal work (plastic dissipation)
x,y: axes of beam co-ordinate system
α: stirrup inclination (with respect to beam axis)
α_M: optimal stirrup inclination
β: inclination of displacement rate relative to yield line normal
γ: angle between yield line and uniaxial compressive stress
δ: relative displacement rate at yield line
η: main reinforcement parameter $F_y/bh\sigma_c$
θ: inclination of diagonal yield line (shear crack)
ν: web effectiveness $\sigma_c^\star/\sigma_c$
ρ: shear reinforcement parameter $1-2\psi$
σ_1,σ_2: principal stresses in concrete (plane stress)
σ_b: uniaxial compressive stress in concrete
σ_c: compressive concrete cylinder strength
$\sigma_c^\star$: effective compressive concrete web strength
σ_x,σ_y: stresses in the x,y-system
τ: nominal shear stress P/bh or pa/h
$\tau^\star$: effective shear strength $V/bh^\star$
τ_M: nominal shear stress for optimal stirrup inclination
τ_{xy}: shear stress in the x,y-system
φ: inclination of uniaxial web compression
ψ: shear reinforcement parameter s_y/σ_c
ψ_0: shear reinforcement parameter for which $\cot\theta = a/h$

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SUMMARY

The load-carrying capacity in shear of simply supported reinforced concrete beams is studied. A solution is presented based on the mathematical theory of plasticity.

Identical upper and lower bounds are found for beams with concentrated loading and vertical or inclined stirrups and for beams with distributed loading and vertical stirrups. The failure mechanism consists of a vertical translation of the central part of the beam with respect to the supports, without any rotation of the beam ends. As the failure criterion for the concrete the square yield locus for plane stress is used. It is shown that with the proposed failure mechanism this predicts shear stresses in the diagonal yield line (crack).

RESUMÉ

Plasticitetsteorien anvendes til bestemmelse af forskydningsbæreevnen af jernbetonbjælker armeret med skrå eller lodrette bøjler.

Sammenfaldende øvre og nedre værdier er fundet for simpelt understøttede bjælker belastet med to symmetriske enkeltkræfter (skrå eller lodrette bøjler) eller med jævnt fordelt belastning (lodrette bøjler). Brudmekanismen består af en ren forskydningsdeformation uden rotation af bjælkeenderne. Som brudbetingelse for betonen benyttes den kvadratiske flydekurve for plan spændingstilstand. Det påvises, at den antagne brudmekanisme indebærer eksistensen af forskydningsspændinger i de skrå brudlinier. De teoretiske bæreevner er sammenlignet med danske og udenlandske bjælkeforsøg.

(Received September 1974, revised March 1975)

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DANMARKS TEKNISKE HØJSKOLE

Structural Research Laboratory

Technical University of Denmark, DK-2800 Lyngby

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