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STABILITY ANALYSIS OF BEAMS AND ARCHES BY ENERGY METHODS

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Stability analysis of beams and arches by energy methods.

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Summary.

Analysis of the stability of curved space beams is based on the potential and the complementary energy theorems. For straight beams examples using the methods for approximate analysis are presented.

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Contents.

Introduction.

In this report, stability criteria for arbitrary curved beams, based on the potential energy and the complementary energy theorems are presented, and some simple examples of their use are given.

The work was carried out partly in connection with the preparation of a textbook on curved beams [1]. It soon became clear that the engineering literature generally curved beams is rather limited, and the literature on their stability is even more limited. Because of this, the theory was basicly reformulated. Some of the results are believed to be new.

A great deal of work of highly theoretical character has been carried out on generally curved beams. See refs. [2] - [10].

The engineering literature was initiated by the increased use of arch bridges, and throughout the thirties, investigations of the stability of arches were reported [11] - [18]. This work has later been continued and the scope of investigation broadened [19] - [29].

Energy methods for stability analysis have been presented in the works [30] - [37].

Derivation of Stability equations.

A structural problem can often be formulated as a variational problem

$$\frac{\partial}{\partial x} \Pi \ (\underset{\sim}{x};\underset{\sim}{p}) \ d\underset{\sim}{x} = 0 \tag{1}$$

where $\underset{\sim}{x}$ is a vector of the unknown displacements or stresses and $\underset{\sim}{p}$ is a vector of prescribed loadings or displacements.

From equation (1) an incremental equation can be obtained

$$\frac{\partial}{\partial \underset{\sim}{x}} \left(\frac{\partial}{\partial \underset{\sim}{x}} \Pi \ (\underset{\sim}{x};\underset{\sim}{p}) \ d\underset{\sim}{x} + \frac{\partial}{\partial \underset{\sim}{p}} \Pi \ (\underset{\sim}{x};\underset{\sim}{p}) d\underset{\sim}{p}\right) \ d\underset{\sim}{x} = 0 \tag{2}$$

When the first term of eq. (2) satisfies the equation

$$\frac{\partial}{\partial \underset{\sim}{x}} \frac{\partial}{\partial \underset{\sim}{x}} \Pi \ (\underset{\sim}{x};\underset{\sim}{p}) \ d\underset{\sim}{x} \ d\underset{\sim}{x} = 0 \tag{3}$$

then $\underset{\sim}{x}$ is no more uniquely defined, and hence equation (3) is equivalent to the Euler Stability criterion, which states that the stability of a structure is lost when there exists an equilibrium configuration infinitely close to the present one for the same loading.

Stability equations for arbitrary curved beams, derived by the potential energy theorem.

In ref. [1] the stability criterion for arbitrary curved beams is derived. The lengthy derivation will not be repeated here, only the basic kinematics.

In figure 1, a cross-section of a beam in the deflected configuration is shown. The cross-section remains plane and is characterized by the unit vectors $\underset{\sim}{e}_1$ and $\underset{\sim}{e}_2$ and the unit normal $\underset{\sim}{e}_3$. The tangent of the system line is $\underset{\sim}{a} = \frac{\partial \underset{\sim}{r}}{\partial \xi} = \underset{\sim}{r}'$. Furthermore we define $a^2 = \underset{\sim}{a} \cdot \underset{\sim}{a}$. In the reference configuration, the above defined quantities are given by capital letters $\underset{\sim}{E}_1$, $\underset{\sim}{E}_2$, $\underset{\sim}{E}_3$, $\underset{\sim}{A}$ and A.

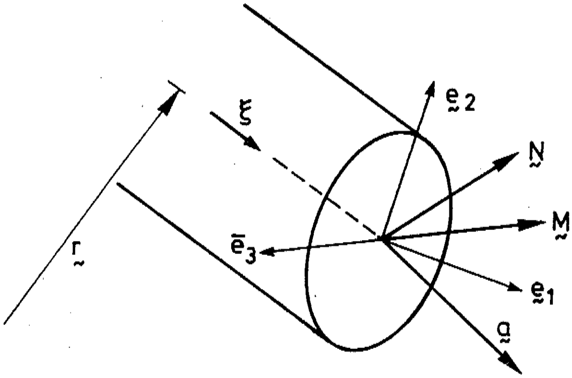

Figure 1.

The deformation of the beam is completely characterized by the strain measures

$$\underset{\sim}{d} = \{\varepsilon_1 \quad \varphi_2 \quad \varphi_3 \quad \tau \quad \kappa_2 \quad \kappa_3\} \tag{4}$$

defined by

$$\left.\begin{aligned}
A\,\varepsilon_1 &= \underset{\sim}{a}\cdot\underset{\sim}{e}_1 - A \\
A\,\varphi_2 &= \underset{\sim}{a}\cdot\underset{\sim}{e}_2 \\
A\,\varphi_3 &= \underset{\sim}{a}\cdot\underset{\sim}{e}_3 \\
A\,\tau &= t-T \quad , \quad t = \underset{\sim}{e}_2'\cdot\underset{\sim}{e}_3 \quad , \quad T = \underset{\sim}{E}_2'\cdot\underset{\sim}{E}_3 \\
A\,\kappa_2 &= k_2-K_2, \quad k_2 = \underset{\sim}{e}_3'\cdot\underset{\sim}{e}_1 \quad , \quad K_2 = \underset{\sim}{E}_3'\cdot\underset{\sim}{E}_1 \\
A\,\kappa_3 &= k_3-K_3, \quad k_3 = \underset{\sim}{e}_1'\cdot\underset{\sim}{e}_2 \quad , \quad K_3 = \underset{\sim}{E}_1'\cdot\underset{\sim}{E}_2'
\end{aligned}\right\} \tag{5}$$

If we restrict our attention to Bernoulli beams, we have $\varphi_2 = \varphi_3 = 0$. The kinematic theory used here is a special case of the more general treatment given by Erichsen and Truesdell [3]. The strain measures defined by equation (5) are very convenient because the work performed by the force $\underset{\sim}{N}$ and moment $\underset{\sim}{M}$ is

$$dW = N_1 d\varepsilon_1 + N_2 d\varphi_2 + N_3 d\varphi_3 + M_1 d\tau$$
$$+ M_2 d\kappa_2 + M_3 d\kappa_3 \qquad (6)$$

where

$$N_i = \underset{\sim}{N} \cdot \underset{\sim}{e}_i \qquad i = 1,2,3$$
$$M_i = \underset{\sim}{M} \cdot \underset{\sim}{e}_i \qquad i = 1,2,3 \qquad (7)$$

If we restrict our attention to an elastic beam having a linear relation between the strain vector $\underset{\sim}{d}$ and the stress vector

$$\underset{\sim}{\Sigma} = \{N_1 \quad N_2 \quad N_3 \quad M_1 \quad M_2 \quad M_3\} \qquad (8)$$

we can write

$$\underset{\sim}{\Sigma} = \underset{\sim}{G}\, \underset{\sim}{d} \qquad (9)$$

The total potential energy of the beam is then

$$\Pi(\underset{\sim}{r}, \underset{\sim}{e}_i) = \int \frac{1}{2} \underset{\sim}{d}^T \underset{\sim}{G}\, \underset{\sim}{d}\, A\, d\xi + \Pi_e \qquad (10)$$

where Π_e is the potential energy of the external loading.

The equilibrium equations can be obtained from eq. (10) by taking the 1st variation

$$\delta\Pi = 0 \qquad (11)$$

and the stability criterion, eq. (3), by taking the 2nd variation

$$\delta^2\Pi = 0 \qquad (12)$$

Using equation (5) we obtain for a Bernoulli beam

$$\delta^2\Pi = \int_{\xi} \{\delta \underset{\sim}{d}^T \underset{\sim}{G}\delta\underset{\sim}{d} + N_1 \frac{a}{A} (\delta\omega_2^2 + \delta\omega_3^2)$$

$$+ M_1 \frac{1}{A} (\delta\omega_2' \delta\omega_3 - \delta\omega_3' \delta\omega_2 - (\delta\omega_2^2 + \delta\omega_3^2) t + \frac{2}{a} (k_3\delta\omega_2 - k_2\delta\omega_3) d\varepsilon_1)$$

$$+ M_2 \frac{1}{A} (\delta\omega_3'\delta\omega_1 - \delta\omega_1'\delta\omega_3 + 2t\delta\omega_1\delta\omega_2 - (\delta\omega_1^2 + \delta\omega_3^2) k_3 + k_3 \delta\omega_2\delta\omega_3 + \frac{2}{a} t\delta\omega_3\delta\varepsilon_1)$$

$$+ M_3 \frac{1}{A} (\delta\omega_1'\delta\omega_2 - \delta\omega_2'\delta\omega_1 + 2t\delta\omega_1\delta\omega_3 - (\delta\omega_1^2 + \delta\omega_2^2) k_3 + k_2 \delta\omega_2\delta\omega_3 - \frac{2}{a} t\delta\omega_3\delta\varepsilon_1)$$

$$M_2' \frac{1}{A} (-\delta\omega_1\delta\omega_3 + \frac{2}{a} \delta\omega_2\delta\varepsilon_1) + M_3' \frac{1}{A} (\delta\omega_1\delta\omega_2 + \frac{2}{a} \delta\omega_3\delta\varepsilon_1)\} A d\xi + \delta^2\Pi_c = 0 \qquad (13)$$

In eq. (13) $\delta\omega_i$ is the virtual rotation of the coordinate system $\underset{\sim}{e}_i$ defined by

$$\left.\begin{aligned} \delta\underset{\sim}{e}_1 &= \delta\omega_3 \underset{\sim}{e}_2 - \delta\omega_2 \underset{\sim}{e}_3 \\ \delta\underset{\sim}{e}_2 &= \delta\omega_1 \underset{\sim}{e}_3 - \delta\omega_3 \underset{\sim}{e}_1 \\ \delta\underset{\sim}{e}_3 &= \delta\omega_2 \underset{\sim}{e}_1 - \delta\omega_1 \underset{\sim}{e}_2 \end{aligned}\right\} \qquad (14)$$

This stability criterion was published in ref. [1] and is believed to be new. It is a general criterion also valid for large prebuckling deformations. In this case, a, t, k_2 are nonlinear functions of the loading intensity and hence, eq. (13) is nonlinear. If we neglect prebuckling deformations, equation (13) is a linear eigenvalue problem, equvivalent to the Rayleigh quotient for straight beams.

For a straight beam loaded in the 1-2 plane we obtain, when neglecting prebuckling deformations, and assuming $\delta\varepsilon_1 = 0$ and $\delta\kappa_3 = 0$,

$$\delta^2\Pi = \int_L \{GI_s \ \delta\tau^2 + EI_2 \ \delta\kappa_2^2 + N_1(\delta\omega_2^2 + \delta\omega_3^2) - 2M_3 \ \delta\omega_1\delta\omega_2'\} dL = 0 \qquad (15)$$

From equation (15), both the Euler buckling load and the torsional buckling load can be obtained.

Stability equations for beams derived by the complementary energy theorem.

A generalized complementary principle is described in ref. [38]. For curved beams it takes, according to ref. [1], the form

$$\Pi_c = \int_\xi \{\tfrac{1}{2} \underset{\sim}{\Sigma}^T \underset{\sim}{G}^{-1} \underset{\sim}{\Sigma} + N_1(E_1-\beta_2\varphi_3+\beta_3\varphi_2)+N_2(E_2+\beta_1\varphi_3-\beta_3\varepsilon_1)$$

$$+ N_3(E_3-\beta_1\varphi_2+\beta_2\varepsilon_1) + M_1\chi_1+M_2\chi_2+M_3\chi_3\} \, A d\xi \qquad (16)$$

where

$$\Delta\underset{\sim}{E}_1 = \underset{\sim}{e}_1-\underset{\sim}{E}_1 \;,\; \Delta\underset{\sim}{E}_2 = \underset{\sim}{e}_2-\underset{\sim}{E}_2 \;,\; \Delta\underset{\sim}{E}_3 = \underset{\sim}{e}_3-E_3 \qquad (17)$$

$$E_1 = \tfrac{1}{2}\Delta\underset{\sim}{E}_1^2 \;,\quad E_2 = \tfrac{1}{2}\Delta\underset{\sim}{E}_1\cdot\Delta\underset{\sim}{E}_2 \;,\quad E_3 = \tfrac{1}{2}\Delta\underset{\sim}{E}_1\cdot\Delta\underset{\sim}{E}_3 \qquad (18)$$

$$\left.\begin{aligned}
\beta_1 &= \tfrac{1}{2}(\Delta\underset{\sim}{E}_2\cdot\underset{\sim}{e}_3 - \Delta\underset{\sim}{E}_3\cdot\underset{\sim}{e}_2)\\
\beta_2 &= \tfrac{1}{2}(-\Delta\underset{\sim}{E}_1\cdot\underset{\sim}{e}_3 + \Delta\underset{\sim}{E}_3\underset{\sim}{e}_1)\\
\beta_3 &= \tfrac{1}{2}(\Delta\underset{\sim}{E}_1\,\underset{\sim}{e}_2 - \Delta\underset{\sim}{E}_2\,\underset{\sim}{e}_1)
\end{aligned}\right\} \qquad (19)$$

$$\left.\begin{aligned}
A\chi_1 &= -(-\tfrac{1}{2}\Delta\underset{\sim}{E}_2'\cdot\Delta\underset{\sim}{E}_3 + \tfrac{1}{2}\Delta\underset{\sim}{E}_2\cdot\Delta\underset{\sim}{E}_3' + \tfrac{1}{2}(\Delta\underset{\sim}{E}_2^2+\Delta\underset{\sim}{E}_3^2)t\\
&\quad - \tfrac{1}{2}\Delta\underset{\sim}{E}_1\cdot\Delta\underset{\sim}{E}_2\,k_2 - \tfrac{1}{2}\Delta\underset{\sim}{E}_1\cdot\Delta\underset{\sim}{E}_3\,k_3)\\
A\chi_2 &= -(\tfrac{1}{2}\Delta\underset{\sim}{E}_1'\cdot\Delta\underset{\sim}{E}_3 - \tfrac{1}{2}\Delta\underset{\sim}{E}_1\cdot\Delta\underset{\sim}{E}_3' - \tfrac{1}{2}\Delta\underset{\sim}{E}_1\cdot\Delta\underset{\sim}{E}_2 t\\
&\quad + \tfrac{1}{2}(\Delta\underset{\sim}{E}_1^2 + \Delta\underset{\sim}{E}_3^2)\,k_2 - \tfrac{1}{2}\Delta\underset{\sim}{E}_2\cdot\Delta\underset{\sim}{E}_3\,k_3)
\end{aligned}\right\} \qquad (20)$$

$$A\chi_3 = -\left(\frac{1}{2}\,\Delta\underset{\sim}{E}_1\;\Delta\underset{\sim}{E}_2' - \frac{1}{2}\,\Delta\underset{\sim}{E}_1'\cdot\Delta\underset{\sim}{E}_2 - \frac{1}{2}\,\Delta\underset{\sim}{E}_1\cdot\Delta\underset{\sim}{E}_3 t - \frac{1}{2}\,\Delta\underset{\sim}{E}_2\cdot\Delta\underset{\sim}{E}_3\;k_2 + \frac{1}{2}\,(\Delta\underset{\sim}{E}_1^2+\Delta\underset{\sim}{E}_2^2)\;k_3\right) \tag{20}$$

The solution of the elastic problem is obtained by

$$\delta\Pi_c\;(\underset{\sim}{\Sigma},\;\underset{\sim}{r},\;\underset{\sim}{e}_i) = 0 \tag{21}$$

where the stresses $\underset{\sim}{\Sigma}$ must satisfy apriori the equilibrium equations.

According to equation (3), we get the stability criterion

$$\delta^2\Pi_c = \int_\xi \{d\underset{\sim}{\Sigma}^T\;\underset{\sim}{G}^{-1}\;d\underset{\sim}{\Sigma} + N_1(\delta\omega_3^2+\delta\omega_2^2-\delta\omega_2\;\delta\varphi_3+\delta\omega_3\delta\varphi_2)$$
$$+ N_2(-\delta\omega_1\delta\omega_2+\delta\omega_1\delta\varphi_3-\delta\omega_3\delta\varepsilon_1)+N_3(-\delta\omega_3\delta\omega_1-\delta\omega_1\delta\varphi_2+\delta\omega_2\delta\varepsilon_1)$$
$$+ M_1(\delta\omega_2'\delta\omega_3-\delta\omega_2\delta\omega_3'-(\delta\omega_2^2+\delta\omega_3^2)T+\delta\omega_1\delta\omega_3\;K_3+\delta\omega_1\delta\omega_2\;K_2)$$
$$+ M_2(\delta\omega_3'\delta\omega_1-\delta\omega_1'\delta\omega_3+\delta\omega_1\delta\omega_3\;T-(\delta\omega_1^2+\delta\omega_3^2)\;K_2+\delta\omega_2\delta\omega_3\;K_3)$$
$$+ M_3(\delta\omega_1'\delta\omega_2-\delta\omega_2'\delta\omega_1+\delta\omega_1\delta\omega_3\;T+\delta\omega_2\delta\omega_3 K_2-(\delta\omega_1^2+\delta\omega_2^2)\;K_3)\}$$
$$A d\xi = 0 \tag{22}$$

when neglecting the prebuckling deformations. It is interesting to notice that equation (22) is valid both for the Timoshenko and Navier theory, because the condition $\delta\varphi_2 = \delta\varphi_3 = 0$ in this formulation is a natural condition which need not be satisfied apriori. However, for a Navier beam, better results can be expected when we apriori assume $\delta\varphi_2 = \delta\varphi_3 = 0$, which will also decrease the number of displacement variables.

For a straight Navier beam loaded in the 1,2 plane, and assuming $\delta\varepsilon_1 = \delta\varphi_2 = \delta\varphi_3 = \delta M_3 = 0$, we get

$$\delta^2\Pi_c = \int_L \{ \frac{1}{GI_v} \delta M_1^2 + \frac{1}{EI_2} \delta M_2^2$$

$$+ N_1 (\delta\omega_3^2+\delta\omega_2^2) + N_2 (-\delta\omega_2\delta\omega_1) + M_3 (\delta\omega_1'\delta\omega_2-\delta\omega_2'\delta\omega_1) \} dL = 0 \quad (23)$$

Using the equilibrium equation $M_3' = - N_2$, and integrating by part we obtain

$$\delta^2\Pi_c = \int_L \{ \frac{1}{GI_s} \delta M_1^2 + \frac{1}{EI_2} \delta M_2^2 + N_1(\delta\omega_3^2 + \delta\omega_2^2)$$

$$- 2 M_3 \delta\omega_1 \delta\omega_2' \} dL = 0 \quad (24)$$

In equation (24) we have neglected boundary terms obtained by the integration. For the most common boundary conditions those terms vanish. In the case of a free end, loaded by a moment M_3, the term does not vanish. However, this case requires more carefull treatment all the same, and the result is dependent on the manner in which the moment M_3 is applied.

It is interesting to notice the similarity between the equations (24) and (15).

Examples.

In order to illustrate the use of energy method for approximate calculation of stability loads, a series of examples are presented in the following. In the examples 1 to 4, the potential energy method is used, and the complementary energy method in examples 5 to 8.

Example 1.a.

The simply supported beam, shown in figure 2, is prevented from rotation at the ends, and is analysed approximately, assuming the following displacements

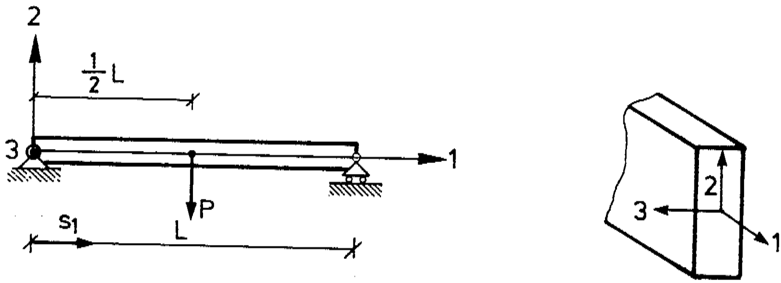

Figure 2.

$$\delta u_3 = c_1 \sin \frac{\pi s_1}{L} \;,\; \delta\omega_1 = c_2 \sin \frac{\pi s_1}{L} \;;$$

which satisfy the kinematic boundary conditions $\delta u_3 = \delta\omega_1 = 0$.
By equation (15) we then get

$$\delta^2\Pi = EI_2\, c_1^2 \left(\frac{\pi}{L}\right)^4 \frac{L}{2} + GI_s\, c_2^2 \left(\frac{\pi}{L}\right)^2 \frac{L}{2} - 2\, c_1 c_2\, \frac{1}{4}\, PL^2\; 0.3513 \left(\frac{\pi}{L}\right)^2 = 0$$

or, using matrix notation,

$$\delta^2\Pi = \underset{\sim}{x}^T (\underset{\sim}{A} - \lambda \underset{\sim}{B})\, \underset{\sim}{x}\, \frac{EI_2}{L} = 0$$

Here $\underset{\sim}{x}$ and λ are defined by

$$\underset{\sim}{x} = \{c_1/L \quad c_2\} \;,\; \lambda = \frac{PL^2}{EI_2} \;,$$

and the matrices $\underset{\sim}{A}$ and $\underset{\sim}{B}$ by

$$\underset{\sim}{A} = \begin{bmatrix} \frac{1}{2}\pi^4 & 0 \\ 0 & \frac{1}{2}\pi^2 \frac{GI_s}{EI_2} \end{bmatrix} \;,\; \underset{\sim}{B} = \begin{bmatrix} 0 & \pi^2\, 0.0878 \\ \pi^2\, 0.0878 & 0 \end{bmatrix}$$

We then obtain the characteristic equation

$$\lambda^2 = \frac{\frac{1}{4}\ \pi^6 GI_s/EI_2}{\pi^4\ 0.0878^2} \text{ or } \lambda = \frac{\pi \quad GI_s/EI_2}{0.1757}$$

The critical load is therefore

$$P = \frac{\pi \sqrt{EI_2 GI_s}}{0.1757\ L^2} = \frac{17.89 \sqrt{EI_2 GI_s}}{L^2}$$

which is 5% greater than the exact result.

Example 1.b.

If the beam in example 1.a is loaded outside the system line, then the loading will contribute to $\delta^2\Pi$.

Let the force act at a distance h above the system line. The additional contribution to the second variation is

$$-Ph(\delta\omega_1)^2_{s_1=L/2} = -Ph\ c_2^2$$

Then we get a modified matrix $\underset{\sim}{B}$ equal to

$$\underset{\sim}{B} = \begin{bmatrix} 0 & \pi^2\ 0.1757 \\ \pi^2\ 0.1757 & 2\ \frac{h}{L} \end{bmatrix}$$

and the following characteristic equation

$$3.0071\ \lambda^2 + \frac{h}{L}\ 19.74\ \lambda - 9.8696\ \frac{GI_s}{EI_2} = 0$$

If we assume $(\frac{h}{L})^2 << 1$, we get the formula

$$P = \frac{17.89\sqrt{EI_2GI_s}}{L^2}\left(1-1.81\frac{h}{L}\sqrt{\frac{EI_2}{GI_s}}\right)$$

The formula given by Timoshenko [40] is

$$P = \frac{16.94\sqrt{EI_2GI_s}}{L^2}\left(1-1.74\frac{h}{L}\sqrt{\frac{EI_2}{GI_s}}\right)$$

Example 1.c.

If we wish to do a more accurate analysis of the case treated in example 1.a,we may assume

$$\delta u_3 = c_1 \sin\frac{\pi s_1}{L} + c_2 \sin 3\frac{\pi s_1}{L}$$

$$\delta\omega_1 = d_1 \sin\frac{\pi s_1}{L} + d_2 \sin 3\frac{\pi s_1}{L}$$

We then find

$$\delta^2\Pi = \underset{\sim}{x}^T(\underset{\sim}{A}-\lambda\underset{\sim}{B})\,\underset{\sim}{x}\,\frac{EI_2}{L} = 0$$

where

$$x = \{c_1/L \quad c_2/L \quad d_1 \quad d_2\} = \{\underset{\sim}{x}_1^T \; \underset{\sim}{x}_2^T\}\ ,\ \lambda = \frac{PL^2}{EI_2}\ ,$$

$$\underset{\sim}{A} = \left\lceil \frac{1}{2}\pi^4 \quad 81/2\,\pi^4 \quad \frac{1}{2}\frac{GI_s}{EI_2}\pi^2 \quad 9/2\,\frac{GI_s}{EI_2}\pi^2 \right\rfloor = \lceil \underset{\sim}{A}_1 \; \underset{\sim}{A}_2 \rfloor$$

$$\underset{\sim}{B} = \begin{bmatrix} 0 & 0 & -0.8668 & 0.2500 \\ 0 & 0 & 2.2496 & -5.7977 \\ -0.8668 & 2.2496 & 0 & 0 \\ 0.2500 & -5.7977 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{B}_1 \\ \underset{\sim}{B}_1^T & \underset{\sim}{0} \end{bmatrix}$$

By eliminating $\underset{\sim}{x}_1$ the eigenvalue problem can be reduced to

$$\underset{\sim}{x}_2^T \left(\frac{1}{\lambda^2} \underset{\sim}{A}_2 - \underset{\sim}{B}_1^T \underset{\sim}{A}_1^{-1} \underset{\sim}{B}_1\right) \underset{\sim}{x}_2 = 0$$

where

$$\underset{\sim}{B}_1^T \underset{\sim}{A}_1^{-1} \underset{\sim}{B}_1 = \begin{bmatrix} 0.01671 & -0.007755 \\ -0.007755 & 0.009805 \end{bmatrix} .$$

The characteristic equation is then

$$\frac{1}{\lambda^4} \frac{GI_s}{EI_2} 219.1 - \frac{1}{\lambda^2} \sqrt{\frac{GI_s}{EI_2}} 0.7905 + 0.0001037 = 0 ,$$

and we obtain

$$P = \frac{16.97 \sqrt{EI_2 GI_s}}{L^2}$$

which is 0.2% greater than the exact result

$$P = \frac{16.94 \sqrt{EI_2 GI_s}}{L^2}$$

Example 1.d.

The same beam is again considered, but including warping rigidity. Eq. (15) is than modified to

$$\delta^2 \Pi = \int_L \{GI_s \ \delta\tau^2 + EI_2 \ \delta\kappa_2^2 + EI_\omega \ \delta\tau'^2$$

$$- 2 M_3 \ \delta\omega_1 \ \delta\omega_2'\} \ dL = 0$$

We assume the same displacements as in example 1.a. The matrix $\underset{\sim}{A}$ is modified to

$$A = \begin{bmatrix} \pi^2 & 0 \\ 0 & \frac{GI_s}{EI_2} + \frac{EI_\omega}{EI_2} \left(\frac{\pi}{L}\right)^2 \end{bmatrix}$$

and the critical load is found to be

$$P = \frac{17.89 \sqrt{EI_2(GI_s+EI_\omega(\frac{\pi}{L})^2)}}{L^2}$$

If,for example

$$EI_\omega = \frac{1}{8} GI_s L^2$$

then

$$P = \frac{26.74 \sqrt{EI_2 GI_s}}{L^2}$$

which is 4.4% greater than the exact result.

Example 1.e.

We close the treatment of the simply supported beam by considering different loading conditions, see figure 3.

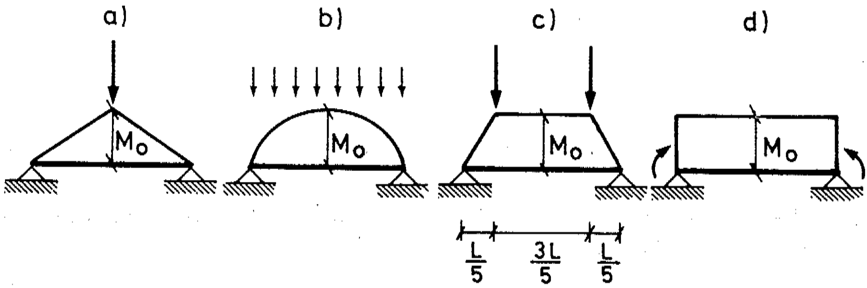

Figure 3.

We use the same displacement assumption as in example 1.a.

$$\delta u_3 = c_1 \sin \frac{\pi s_1}{L} , \quad \delta\omega_1 = c_2 \sin \frac{\pi s_1}{L}$$

and get

$$\delta^2 \Pi = \underset{\sim}{x}^T \, (\underset{\sim}{A} - \lambda \, \underset{\sim}{B}) \, \underset{\sim}{x} \, \frac{EI_2}{L} = 0$$

where $\underset{\sim}{x} = \{c_1/L \quad c_2\}$, $\quad \lambda = \dfrac{M_o L}{EI_2}$,

$$\underset{\sim}{A} = \begin{bmatrix} \frac{1}{2}\,\pi^4 & 0 \\ 0 & \frac{1}{2}\,\frac{GI_s}{EI_2}\,\pi^2 \end{bmatrix}$$

and

$$\underset{\sim}{B} = \begin{bmatrix} 0 & \pi^2 B \\ \pi^2 B & 0 \end{bmatrix} , \quad B = \frac{1}{\pi^2 M_o} \int_L M \delta\omega_1 \; \delta\omega_2' \; dL$$

The characteristic equation is

$$\lambda = \frac{\frac{1}{2} \pm \sqrt{\frac{GI_s}{EI_2}}}{B} ,$$

and the critical moment

$$M_o = \frac{\pi \sqrt{EI_2 GI_s}}{2BL}$$

In the table below the value of M_o is listed for the loading cases shown in figure 3.

CASE	a	b	c	d
$\dfrac{M_o L}{\pi \sqrt{EI_2 GI_s}}$	1.42	1.15	1.02	1.000

Example 2.

A simply supported beam is elastically supported against rotation at one end, see figure 4.

We assume

$$\delta u_3 = c_1 \sin \frac{\pi s_1}{L} + c_2 \sin \frac{\pi s_1}{L}$$

$$\delta \omega_1 = d_1 \sin \frac{\pi s_1}{2L} + d_2 \sin \frac{\pi s_1}{L}$$

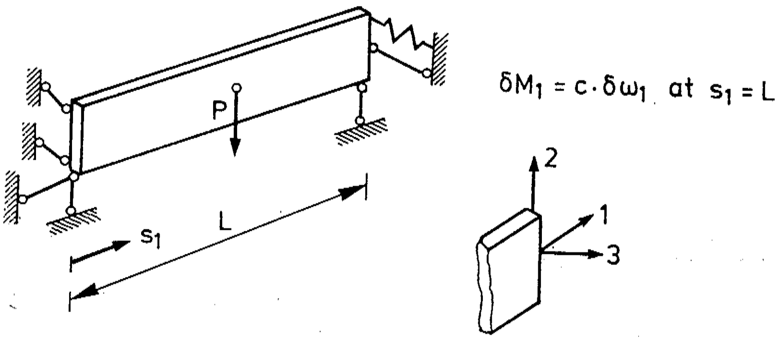

Figure 4.

We then get from eq. (15), adding the second variation of the energy of the spring $c\delta\omega_1^2$

$$\delta^2\Pi = \underset{\sim}{x}^T (\underset{\sim}{A} - \lambda \underset{\sim}{B}) \underset{\sim}{x} \frac{EI_2}{L} = 0$$

Here $\underset{\sim}{x} = \{c_1/L \quad c_2/L \quad d_1 \quad d_2\}$,

$$\underset{\sim}{A} = \begin{bmatrix} \frac{1}{2}\pi^4 & 0 & 0 & 0 \\ 0 & 8\pi^4 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\frac{GI_s}{EI_2}(\frac{\pi}{2})^2 + \frac{c \cdot L}{EI_2} & 1.0472\frac{GI_s}{EI_2} \\ 0 & 0 & 1.0472\frac{GI_s}{EI_2} & \frac{1}{2}\frac{GI_s}{EI_2}\pi^2 \end{bmatrix}$$

and

$$\underset{\sim}{B} = \begin{bmatrix} 0 & 0 & -0.6824 & -0.8668 \\ 0 & 0 & 0.6868 & 0 \\ -0.6824 & 0.6868 & 0 & 0 \\ -0.8668 & 0 & 0 & 0 \end{bmatrix}$$

The critial load can be found by calculations similar to those of example 1.b. In the table the critical loads for different values of the spring stiffness are listed.

cL/GI_s	0	3	6	30	300	∞
$PL^2/\sqrt{EI_2GI_s}$	10.81	15.00	16.08	17.43	17.84	17.89

Example 3.

In this example a continuous beam is analysed, see figure 5. It is loaded in the middle of the left span by a single force, and the prebuckling bending moment is as shown on the figure. The beam is not prevented from rotation at the middle support. The kinematic boundary conditions are

$$\delta u_3 = 0 \quad \text{for} \quad s_1 = 0,\ L,\ 2L$$

and

$$\delta \omega_1 = 0 \quad \text{for} \quad s_1 = 0,\ 2L$$

We assume the displacement field

$$\delta u_3 = c_1 \sin \frac{\pi s_1}{L} + c_2 \sin 2 \frac{\pi s_1}{L} ,$$

$$\delta \omega_1 = d_1 \sin \frac{\pi s_1}{2L} + d_2 \sin \frac{\pi s_1}{L}$$

and obtain

$$\delta^2 \Pi = \underset{\sim}{x}^T (\underset{\sim}{A} - \lambda \underset{\sim}{B}) \underset{\sim}{x} \frac{EI_2}{L} = 0$$

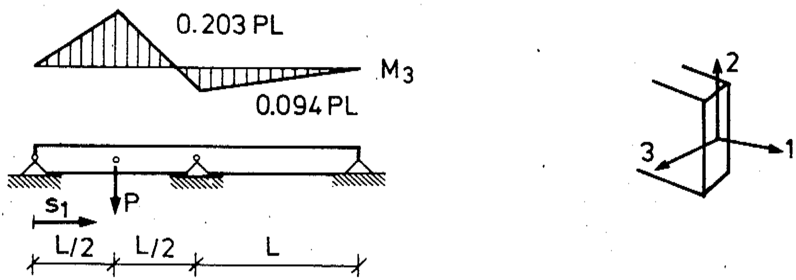

Figure 5

Here

$$\underset{\sim}{x} = \{c_1/L \quad c_2/L \quad d_1 \quad d_2\} \ ,$$

$$\lambda = \frac{PL^2}{EI_2} \ ,$$

$$\underset{\sim}{A} = \begin{bmatrix} \pi^4 & 0 & 0 & 0 \\ 0 & 16\pi^4 & 0 & 0 \\ 0 & 0 & \frac{1}{4}\pi^2 \frac{GI_s}{EI_2} & 0 \\ 0 & 0 & 0 & \pi^2 \frac{GI_s}{EI_2} \end{bmatrix}$$

and

$$\underset{\sim}{B} = \begin{bmatrix} 0 & 0 & -0.6948 & -0.4196 \\ 0 & 0 & 0.6859 & -0.6709 \\ -0.6948 & 0.6859 & 0 & 0 \\ -0.4196 & -0.6709 & 0 & 0 \end{bmatrix}$$

Proceeding as in example 1.c, we obtain the result

$$P = \frac{20.96 \sqrt{EI_2 GI_s}}{L^2} \ .$$

If a more simple approximation was made by assuming $c_2 = d_2 = 0$ we would get

$$P = \frac{22.33 \sqrt{EI_2 GI_s}}{L^2} \ ,$$

which is only 7% greater than the result obtained by the more accurate analysis.

Example 4.

As a last example using the potential energy method, an arch loaded by end moments as indicated in figure 6 is considered.

The arch is supported such that

$$\delta u_3 = \delta\omega_1 = 0 \quad \text{for} \quad s_1 = 0, L \ .$$

The following displacements are assumed

$$\delta u_3 = c_1 \sin \frac{\pi s_1}{L} \ ,$$

$$\delta\omega_1 = d_1 \sin \frac{\pi s_1}{L} \ .$$

Furthermore we have

$$M_3 = M \text{ and } k_1 = k_2 = M_1 = M_2 = N_1 = 0 \ .$$

Equation (13) is then reduced to

$$\delta^2\Pi = \int_L \{EI_2 \ \delta\kappa_2^2 + GI_s \ \delta\tau^2$$

$$+ M_3(\delta\omega_1'\delta\omega_2 - \delta\omega_2'\delta\omega_1 - (\delta\omega_1^2 + \delta\omega_2^2) \ k_3\} \quad dL = 0$$

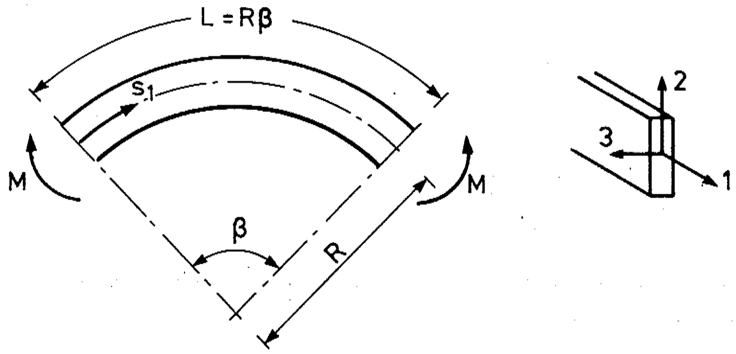

Figure 6.

where

$$\delta\tau = \delta\omega_1' - k_3\ \delta\omega_2\ ,$$

$$\delta\kappa_2 = \delta\omega_2' + k_3\ \delta\omega_1\ ,$$

$$\delta\omega_2 = -\ \delta u_3'$$

and we obtain

$$\delta^2\Pi = EI_2\ \frac{L}{2}((\frac{\pi}{L})^4\ c_1^2 + k_3^2\ d_1^2 + 2\ k_3\ (\frac{\pi}{L})^2\ c_1\ d_1)$$

$$+ GI_s\ \frac{L}{2}\ ((\frac{\pi}{L})^4\ c_1^2 + k_3^2\ d_1^2 + 2\ k_3\ (\frac{\pi}{L})^2\ c_1\ d_1)$$

$$- M\ \frac{L}{2}\ (2(\frac{\pi}{L})^2\ c_1\ d_1 + ((\frac{\pi}{L})^2\ c_1^2 + d_1^2)\ k_3) = 0$$

or

$$\delta^2\Pi = \underset{\sim}{x}^T\ (\underset{\sim}{A} - \lambda\ \underset{\sim}{B})\ \underset{\sim}{x}\ \frac{1}{2}\ \frac{EI}{L} = 0$$

where

$$\underset{\sim}{x} = \{c_1/l\quad d_1\}\quad ,\quad \lambda = \frac{ML}{EI_2}\ ,$$

$$\underset{\sim}{A} = \begin{bmatrix} \pi^4 + k_3^2\ L^2\ \frac{GI_s}{EI_2}\ \pi^2 & \pi^2\ k_3L + k_3L\ \frac{GI_s}{EI_2}\ \pi^2 \\ \pi^2\ k_3L + k_3L\ \frac{GI_s}{EI_2}\ \pi^2 & k_3^2L^2 + \frac{GI_s}{EI_2}\ \pi^2 \end{bmatrix}$$

and

$$\underset{\sim}{B} = \begin{bmatrix} \pi^2\ k_3L & \pi^2 \\ \pi^2 & k_3L \end{bmatrix}$$

The characteristic equation is

$$\pi^2 ((k_3L)^2 - \pi^2)\ \lambda^2 + \pi^2 (1 + \frac{GI_s}{EI_2})\ (\pi^2 - (k_3L)^2)\ \lambda$$
$$+ \pi^2 \frac{GI_s}{EI_2} (\pi^2 - (k_3L)^2)^2 = 0 .$$

After some calculations we obtain

$$M = \frac{k_3}{2} [(EI_2+GI_s) \pm \sqrt{(EI_2-GI_s)^2 + 4EI_2GI_s \frac{\pi^2}{(k_3L)^2}}]$$

In this formula the prebuckling deformations can be taken into account by calculating k_3 as a function of M. If prebuckling deformations are neglected we have

$$k_3 = K_3 = \frac{1}{R} = \frac{\beta}{L} .$$

Example 5.a.

In the last examples, we shall illustrate the use of the complementary energy method. The first example is the simply supported beam already treated in example 1.a by the potential energy method.

We assume the same displacement field as in example 1.a

$$\delta u_3 = c_1 \sin \frac{\pi s_1}{L} , \quad \delta\omega_1 = c_2 \sin \frac{\pi s_1}{L}$$

From the equilibrium equations we obtain the bending moments

$$\delta M_1 = \frac{P}{2} (c_1 - \delta u_3) + \frac{P}{2} s_1 \delta u_3' , \qquad s_1 < \frac{L}{2}$$

$$\delta M_2 = \frac{P}{2} s_1 \delta\omega_1 , \qquad s_1 < \frac{L}{2}$$

If these are introduced into equation (23), we get

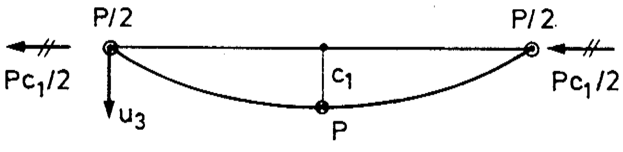

Figure 7.

$$\delta^2\Pi_c = 2\int_0^{L/2}\{ \frac{P^2}{4} s_1^2 c_2^2 \sin^2(\frac{\pi s_1}{L}) \frac{1}{EI_2}$$

$$+ \frac{P^2}{4} c_1^2 (1-\sin(\frac{\pi s_1}{L}) + s_1 \frac{\pi}{L} \cos(\frac{\pi s_1}{L}))^2 \frac{1}{GI_s}$$

$$- \frac{P}{2} s_1 \, 2 c_1 c_2 \sin(\frac{\pi s_1}{L}) (\frac{\pi}{L})^2 \sin(\frac{\pi s_1}{L})\} ds_1$$

$$= \underset{\approx}{x}^T \{\underset{\approx}{A} - \lambda \underset{\approx}{B}\} \underset{\approx}{x} \frac{P^2L^3}{GI_s} = 0 .$$

Here

$$\underset{\approx}{x} = \{c_1/L \quad c_2\} \quad , \quad \lambda = \frac{GI_s}{PL^2} ,$$

$$\underset{\approx}{A} = \begin{bmatrix} 0.154 & 0 \\ 0 & 0.0167 \frac{GI_s}{EI_2} \end{bmatrix}$$

and

$$\underset{\approx}{B} = \begin{bmatrix} 0 & 0.867 \\ 0.867 & 0 \end{bmatrix}$$

The critical load is obtained by the equation

$$|\underset{\sim}{A}-\lambda\underset{\sim}{B}| = 0 \ ,$$

which gives

$$P = \frac{GI_s}{\lambda L^2} = \frac{0.867 \ \sqrt{EI_2 GI_s}}{\sqrt{0.0167 \cdot 0.154} \ L^2} = 17.08 \ \frac{\sqrt{EI_2 GI_s}}{L^2} \ .$$

The result obtained is 1% greater than the exact result. In example 1.a we obtained an error of 5%, using the same displacement assumptions.

Example 5.b.

The advantage of the complementary energy method is that very simple displacement fields can be used. All that is required is that the kinematic boundary conditions are satisfied and that the functional $\delta^2\Pi_c$ can be calculated. In our case all displacement fields are permissible, as long as the term

$$\int M_3 \ \delta\omega_1 \delta\omega_2' \ ds$$

can be computed.

We thus choose the displacement field indicated in figure 8

$$\delta u_3 = c_1 \ \frac{2s_1}{L} \ , \qquad s_1 < \frac{L}{2}$$

$$\delta\omega_1 = c_2 \ \frac{2s_1}{L} \ , \qquad s_1 < \frac{L}{2}$$

In order to write down $\delta\omega_2'$ we introduce the Dirac delta distribution as illustrated in the figure.

As in the previous example, δM_1 and δM_2 are obtained from the equilibrium equations

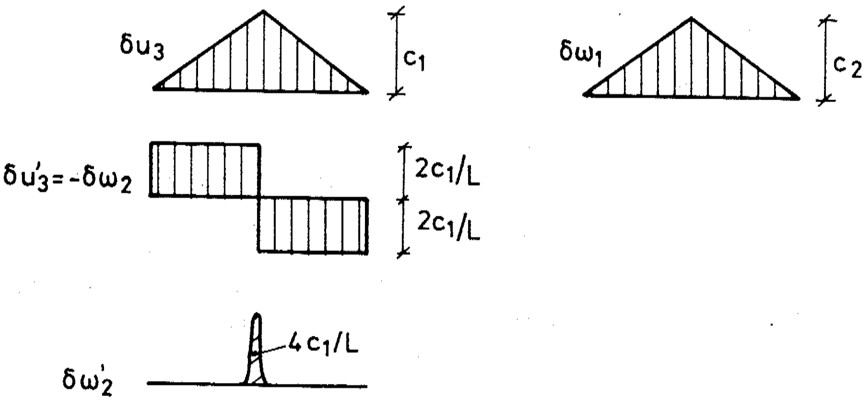

Figure 8.

$$\delta M_1 = \frac{P}{2}\,(c_1 - \delta u_3) + \frac{P}{2}\,s_1\,\delta u_3' \,, \qquad s_1 < \frac{L}{2}$$

$$\delta M_2 = \frac{P}{2}\,s_1\,\delta\omega_1 \,, \qquad s_1 < \frac{L}{2}$$

and we obtain

$$\delta^2\Pi_c = \underset{\sim}{x}^T\,\{\underset{\sim}{A}-\lambda\underset{\sim}{B}\}\,\underset{\sim}{x}\,\frac{P^2L^3}{GI_s} = 0 \,.$$

Here

$$\underset{\sim}{x} = \{c_1/L \quad c_2\} \,, \; \lambda = \frac{GI_s}{PL^2} \,,$$

$$\underset{\sim}{A} = \begin{bmatrix} 0.250 & 0 \\ 0 & 0.0125\,\dfrac{GI_s}{EI_2} \end{bmatrix}$$

and

$$\underset{\sim}{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The critical load is thus

$$P = \frac{\sqrt{EI_2 GI_2}}{\sqrt{0.25 \cdot 0.0125}\ L^2} = 17.89\ \frac{\sqrt{EI_2 GI_s}}{L^2},$$

which,by chance,is the same result as obtained in example 1.a, with a more accurate displacement assumption.

Example 6.

A simply supported beam is only prevented from rotation at one end.

We then may assume the displacement field

$$\delta u_3 = c_1 \sin \frac{\pi s_1}{L}$$

$$\delta \omega_1 = d_1 \sin \frac{\pi s_1}{2L}$$

The bending moments, see figure 9, are found to be

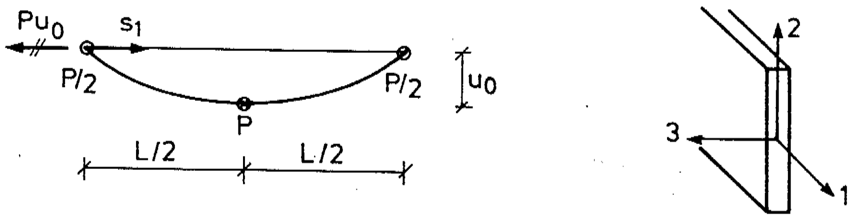

Figure 9.

$$\delta M_1 = \frac{P}{2}\,(2c_1-\delta u_3) + \frac{P}{2}\,s_1\,\delta u_3' , \qquad s_1 < \frac{L}{2}$$

$$\delta M_1 = \frac{P}{2}\,\delta u_3 + \frac{P}{2}\,s_1^*\,\delta u_3' , \quad s_1^* < \frac{L}{2} , \quad s_1^* = L - s_1$$

$$\delta M_2 = \frac{P}{2}\,s_1\,\delta\omega_1 , \qquad s_1 < \frac{L}{2}$$

$$\delta M_2 = \frac{P}{2}\,s_1^*\,\delta\omega_1 , \qquad s_1^* < \frac{L}{2} , \quad s_1^* = L - s_1$$

Introducing these displacements and moments into $\delta^2\Pi_c$ we obtain

$$\delta^2\Pi_c = \underset{\sim}{x}^T(\underset{\sim}{A}-\lambda\underset{\sim}{B})\,\underset{\sim}{x}\,\frac{P^2L^3}{GI_s} = 0 .$$

Here

$$\underset{\sim}{x} = \{c_1/L \quad c_2\} , \qquad \lambda = \frac{GI_s}{PL^2} ,$$

$$\underset{\sim}{A} = \begin{bmatrix} 0.404 & 0 \\ 0 & 0.0105\,\dfrac{GI_s}{EI_2} \end{bmatrix}$$

and

$$\underset{\sim}{B} = \begin{bmatrix} 0 & 0.694 \\ 0.694 & 0 \end{bmatrix}$$

The critical load is then

$$P = \frac{0.694\sqrt{EI_2GI_s}}{\sqrt{0.404\cdot 0.0105}\;L^2} = 10.66\,\frac{\sqrt{EI_2GI_s}}{L^2}$$

In example 2, we obtained the value 10.81, using the potential energy method and 4 modes.

Example 7.a.

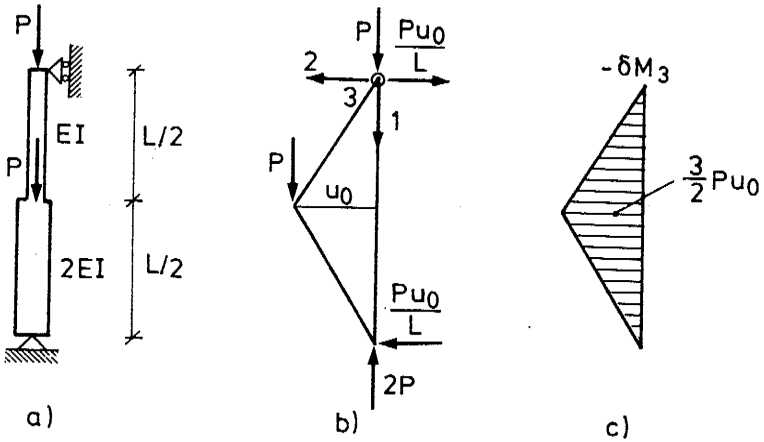

Figure 10.

A simply supported column with variable stiffness is loaded by two equal forces P, as indicated in figure 10 a. Considering only buckling in the 1-2 plane ,we obtain the complementary energy functional

$$\delta^2 \Pi_c = \int_L \{\frac{1}{EI_3} \delta M_3^2 + N_1 \delta\omega_3^2\} dL = 0 .$$

Assuming the simple displacement field in figure 10.b, we obtain the moment field in figure 10.c and

$$\delta^2 \Pi_c = \frac{L/2}{EI} (\frac{3}{2} Pu_o)^2 \frac{1}{3} + \frac{L/2}{2EI} (\frac{3}{2} Pu_o)^2 \frac{1}{3}$$

$$- P L/2 (\frac{2u_o}{L})^2 - 2 PL/2 (\frac{2u_o}{L})^2$$

$$= \frac{P^2 u_o^2 L}{EI} \frac{9}{16} - \frac{Pu_o^2}{L} 6 = 0 .$$

The critical load is then

$$P = 10.67 \frac{EI}{L^2} .$$

In reference [38], the same column was analysed using stability functions, and the result

$$P = 8.96 \frac{EI}{L^2}$$

was obtained. The result obtained here by the very simple analysis has an error of 19%.

Example 7.b.

The same example is treated using a more accurate displacement assumption, see figure 11.a.

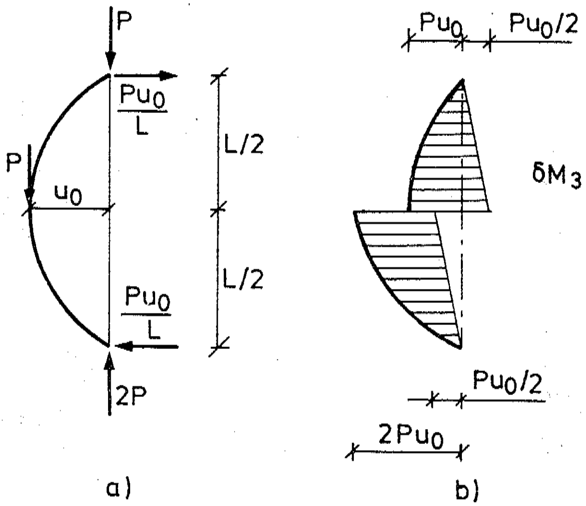

Figure 11.

We obtain the bending moments of figure 11.b, and

$$\delta^2\Pi_c = \frac{P^2u_oL}{EI}\,\frac{7}{8} - \frac{Pu_o^2}{L}\,8 = 0\ ,$$

from which we obtain the critical load

$$P = 9.14\,\frac{EI}{L^2}$$

which is only 2% greater than the exact result.

Example 8.

In order to give a simple illustration of the use of the complementary energy method for statically indeterminate structures, we include the column shown in figure 12.a.

We choose the simple displacement field shown in figure 12.b and choose a spring moment M_o. The spring stiffness is c, and the contribution of the spring to $\delta^2\Pi_c$ is $\frac{1}{c}\,P^2M_o^2$. We then obtain

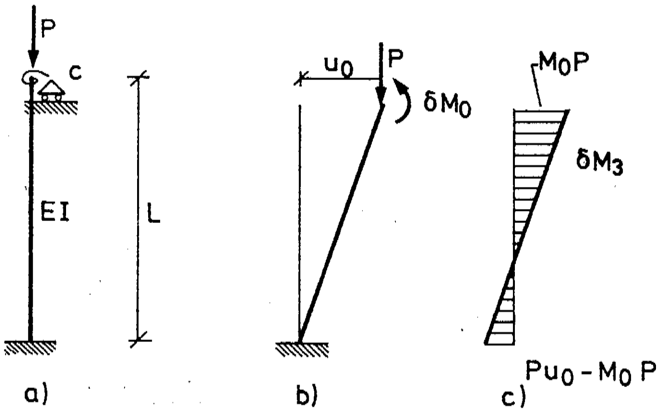

Figure 12.

$$\delta^2\Pi_c = \frac{1}{3}\frac{L}{EI}\left(P^2 u_o^2 + 3P^2 M_o^2 - 3P^2 u_o M_o\right) + M_o^2 P^2 \frac{1}{c}$$

$$- PL \left(\frac{u_o}{L}\right)^2$$

$$= \underset{\sim}{x}^T \left(\underset{\sim}{A} - \lambda \underset{\sim}{B}\right) \underset{\sim}{x} \frac{P^2 L^3}{EI} = 0 \ .$$

Here

$$\underset{\sim}{x} = \{u_o/L \quad M_o/L\} \ , \qquad \lambda = \frac{EI}{PL^2} \ ,$$

$$\underset{\sim}{A} = \begin{bmatrix} 1/3 & -1/2 \\ -1/2 & 1 + \frac{EI}{cL} \end{bmatrix}$$

and $$\underset{\sim}{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The characteristic equation is

$$(1/3 - \lambda)\left(1 + \frac{EI}{cL}\right) = \frac{1}{4} \ ,$$

from which we obtain the critical load

$$P = 12 \frac{1 + \frac{EI}{cL}}{1 + 4\frac{EI}{cL}} \frac{EI}{L^2}$$

In the limiting cases we have

$$c \to \infty \qquad P = 12 \frac{EI}{L^2} \ ,$$

$$c \to 0 \qquad P = 3 \frac{EI}{L^2} \ .$$

Both results have an error of 21%, but the qualitative effect of the spring is obtained with a very simple displacement field.

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