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EFFECT OF MAIN STEEL STRENGTH ON THE SHEAR CAPACITY OF REINFORCED CONCRETE BEAMS WITH STIRRUPS

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EFFECT OF MAIN STEEL STRENGTH ON THE SHEAR CAPACITY OF REINFORCED CONCRETE BEAMS WITH STIRRUPS

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Foreword

The bulk of the research reported was carried out at the Structural Research Laboratory, Copenhagen, the paper being completed at the Engineering Department, Cambridge. The assistance of the staff of the latter in the preparation of the manuscript is gratefully acknowledged.

A condensed version of the present report has been accepted for publication in the "Archiwum Inzynierii Ladowej" ("Archive of Civil Engineering") in the course of 1979.

Lyngby, May 1979

M. W. BRAESTRUP

Summary

Plastic analysis is applied to shear in beams. Steel and concrete are idealized as rigid, perfectly plastic materials, the latter with the modified Coulomb failure criterion and associated flow rule. Upper bound solutions are derived for beams with vertical or inclined stirrups, subjected to concentrated or distributed loading. The failure mechanism may involve yielding of the main reinforcement. The analysis shows the shear strength to be higher for distributed than for point loading, and generally inclined stirrups are more efficient than vertical. Comparison with lower bounds shows the solutions to be exact for beams with strong main reinforcement or with no stirrups. In these cases, excellent agreement with test results is also found.

Resumé

Plasticitetsteorien anvendes på forskydning i jernbetonbjælker. Materialerne antages stive, idealt plastiske, for betonens vedkommende med Coulombs modificerede brudbetingelse (kvadratisk flydekurve) og den associerede flydelov (normalitetsbetingelsen). Øvreværdiløsninger udledes for bjælker med lodrette eller skrå bøjler, påvirket af koncentreret eller jævnt fordelt belastning. Brudmekanismen idebærer mulighed for flydning af hovedarmeringen. Analysen viser at forskydningsstyrken er større for fordelt end for koncentreret belastning og at skrå bøjler normalt er mere effektive end lodrette. Sammenligning med nedreværdier viser at løsningerne er eksakte for bjælker med stærk hovedarmering eller uden forskydningsarmering. I disse tilfælde findes også udmærket overensstemmelse med forsøgsresultater.

EFFECT OF MAIN STEEL STRENGTH ON THE SHEAR CAPACITY OF REINFORCED CONCRETE BEAMS WITH STIRRUPS

INTRODUCTION

The strength of reinforced concrete beams subjected to shear is a simple problem of structural mechancis which for a long time has defied solution. In recent years, however, considerable progress has been made through the application of the classical theory of plasticity. The basic idea is to regard the concrete as a perfectly plastic Coulomb-material with a small (in this case vanishing) tensile strength. The limited ductility of concrete in compression is accounted for by reducing the strength by an empirical effectiveness factor. Using this simple material model, problems of shear in concrete are open to rational treatment by the straightforward methods of limit analysis which have proven their efficiency in connection with flexural problems (yield line theory, yield hinge methods, etc.).

At first glance, it appears highly questionable to describe a material like concrete as perfectly plastic. Rather than engage in a futile discussion regarding the validity of such an assumption, we propose that the description be judged upon its merits, and the theory does furnish some surprisingly accurate predictions of the behaviour of concrete structures subjected to shear.

Another advantage of this approach is that it is a general theory, unifying the analysis of shear in beams (deep or slender, with or without stirrups) with other problems such as punching shear of slabs and shear in joints and corbels, cf. NIELSEN & al. [1], BRAESTRUP & al. [2].

Shear in beams were treated by NIELSEN & BRAESTRUP [3], who derived coinciding upper and lower bounds for beams with point loading and vertical or inclined stirrups and for beams with distributed loading and vertical stirrups. The failure mechanism consists of a diagonal yield line with a vertical displacement rate. The corresponding stress distribution is an inclined compression field in the web concrete and yielding of the stirrups. The solution is equivalent with the classical truss analogy, except that the strut inclination is variable, the optimal inclination - corresponding to failure - depending upon the amount of shear reinforcement. In most cases of practical interest, it is considerably flatter than the classical value of 45°. The concept of variable strut inclination has proven extremely useful in the design of stirrup reinforcement, GROB & THUERLIMANN [4], COLLINS [5], NIELSEN & al. [6]. It has recently been introduced into the CEB-FIP Model Code for structural concrete [7].

The solutions derived in reference [3] are only valid provided the main reinforcement is sufficiently strong to constrain the deformation to be vertical, i.e. the tensile force should be below yield. The purpose of the present paper is to consider instances where this is no longer the case. Upper bound solutions are presented for such beams, assuming a failure mechanism where the displacement rate makes an angle with the beam normal. At the same time, the analysis is extended to cover the case of distributed loading on beams with inclined stirrups.

The basic concepts and assumptions are introduced in Section 1. In Sections 2 through 5, each of the four loading cases are considered. Upper bound solutions are derived in one sub-section, and discussed and compared with known lower bound solutions and available test results in the next. Finally, all the solutions are summarized in Section 6. Some of the expressions are quite complicated, and would presumably not be of much use in design. It is possible, however, to draw some general conclusions concerning the effect of stirrup inclination and type of loading.

NOTATIONS

a : Length of shear span

b : Width of beam web

f_c : Compressive strength of concrete

f_y : Tensile yield stress of stirrups

h : Shear depth of beam

p : Distributed load per unit area

r : Shear reinforcement ratio

T_y : Yield strength of main reinforcement

V : Shear load

v : Relative velocity in yield line

α : Inclination of relative velocity (Figure 2.1)

α^* : Inclination of relative velocity (Figure 1.2)

β : Inclination of yield line (Figure 2.1)

γ : Inclination of stirrups (Figure 3.1)

η : Auxiliary parameter (Equation (5.3))

θ : Inclination of concrete compression (Figure 2.2a)

κ : Reinforcement parameter (Equation $(3.3)_4$)

λ : Shear span ratio ($\lambda=a/h$)

μ : Reinforcement parameter (Equation $(3.3)_2$)

$\bar{\mu}$: Reinforcement parameter (Equation $(3.3)_3$)

ν : Concrete effectiveness factor

ρ : Reinforcement parameter (Equation $(3.3)_1$)

σ_a : Tensile stress in stirrups

τ : Ultimate shear stress

Φ : Main reinforcement degree ($\Phi=T_y/bhf_c$)

ψ : Shear reinforcement degree ($\psi=rf_y/f_c$)

1. BASIC ASSUMPTIONS

Consider a simply supported beam of web width b subjected to shear. The shear span (distance from point of maximum moment to nearest support) is denoted a. The reinforcement consists of longitudinal, main reinforcing bars and equidistant stirrups, vertical or inclined. A plastic analysis of the shear capacity of the beam may be based upon the assumptions listed below:

(a) The main reinforcement at the bottom of the beam and the compression zone at the top act as stringers, carrying a tensile force T and a compressive force C, respectively. The beam depth h is measured as the distance between the stringers.

The stirrups are able to resist axial forces only. The action of the web reinforcement is described by a normal stress $r\sigma_a$, per unit area perpendicular to the stirrups, r being the shear reinforcement ratio and σ_a the tensile stress in the stirrups.

The web concrete is in a state of plane stress.

(b) The stringers and the stirrups are rigid, perfectly plastic. The yield strength of the tensile stringer is T_y. The yield stress of the stirrup steel is f_y.

(c) The concrete of the web is rigid, perfectly plastic with the modified Coulomb failure criterion as yield condition and the associated flow rule (normality condition). The tensile concrete strength is zero and the compressive strength is $f_c^* = \nu f_c$, where f_c is the cylinder strength and ν is an effectiveness factor.

The assumption of rigid-perfect plasticity means that elastic deformations and work-hardening effects are neglected in the analysis. By assumption (a), we further neglect any dowel action of the reinforcement and shear in the compression zone. The stirrup spacing is assumed to be sufficiently small to permit a continuous distribution of the stirrup forces. The shear reinforcement ratio is defined as the area of steel per unit area of a section perpendicular to the stirrups. Hence:

$$r = \frac{A_s}{bc\sin\gamma} \tag{1.1}$$

where A_s is the cross-sectioned steel area per stirrup, γ is the stirrup inclination, and c is the stirrup spacing along the beam axis.

In the case of plane stress, the modified Coulomb criterion with a zero tension cut-off reduces to the so-called square yield locus, sketched on Figure 1.1. The principal stresses are denoted σ_1 and σ_2. Deformations are only possible for stress states (σ_1, σ_2) which are plotted as points on the yield locus. According to the associated flow rule, the ratio between the principal strain rates ε_1 and ε_2 is such that the vector $(\varepsilon_1, \varepsilon_2)$ is an outwards directed normal to the yield locus at the corresponding point (σ_1, σ_2). At a corner, the vector $(\varepsilon_1, \varepsilon_2)$ is required to be situated between the normals to the adjacent faces of the locus (cf. Figure 1.1).

In reality, concrete is not a perfectly plastic material. The behaviour in tension is almost brittle, therefore it is reasonable to neglect the tensile strength. In compression, the ductility of concrete is fairly limited, which means that the redistribution of stresses, assumed in plasticity, can only take place at the expense of losing strength. This suggests the introduction of the effectiveness factor ν as an empirical measure of concrete ductility. The value of ν must be assessed by comparison with experimental evidence.

. A yield line in concrete is a kinematical discontinuity separating the body into two rigid parts. One part is moving relative to the other with the velocity v inclined at the angle α^* to the yield line (Figure 1.2b). The discontinuity is a mathematical idealization of a narrow deforming zone (Figure 1.2a). According to the flow rule, the strain rate state in the zone can only be produced by the stress state $(\sigma_1, \sigma_2) = (o, -f_c^*)$ corresponding to the lower right corner of the yield locus (Figure 1.1). Hence the rate of internal work dissipated per unit area of the yield line is given by (cf. e.g. reference [2]):

$$W_c = \frac{1}{2} v f_c^* \, (1-\sin\alpha^*) \qquad (1.2)$$

valid for $-\pi/2 \leq \alpha^* \leq \pi/2$

The principal directions of stresses and strain rates are indicated on Figure 1.2b. The first principal axis bisects the angle between the relative velocity vector and the yield line normal.

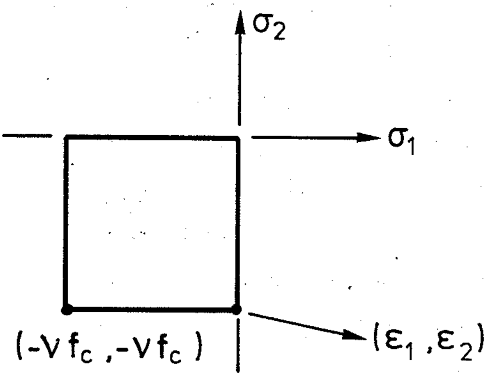
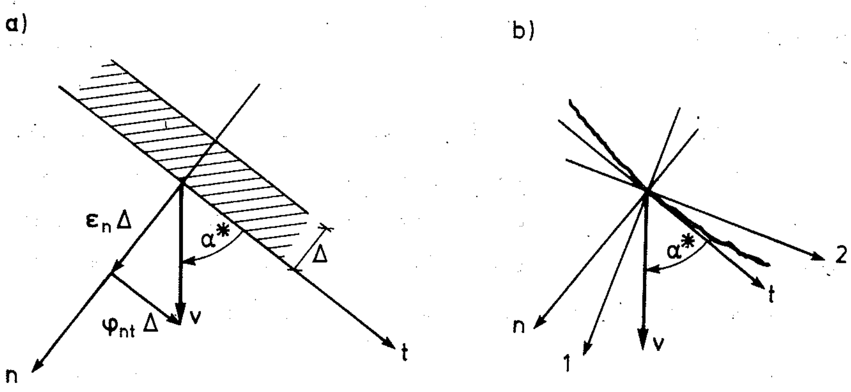


Fig. 1.1 : Square yield locus for concrete in plane stress

Fig. 1.2 : Yield line in plain concrete
a) Narrow deforming zone
b) Kinematical discontinuity



2. CONCENTRATED LOADING, VERTICAL STIRRUPS

2.1 Upper bound solutions

Figure 2.1 shows a shear span of a beam subjected to the shear force V. We assume a failure mechanism consisting of a single yield line inclined at the angle β to the beam axis. The relative velocity is v at the angle α to the beam normal. The rate of internal work dissipated in the mechanism is:

$$(2.1) \quad W_I = rf_y \cos\beta \frac{bh}{\sin\beta} v\cos\alpha + \tfrac{1}{2} v f_c^* \left[1 - \cos(\beta-\alpha) \right] \frac{bh}{\sin\beta} + T_y v\sin\alpha$$

where we have used equation (1.2). The ranges of the variables α and β are:

$$(2.2) \quad \alpha \geq o \qquad \text{and} \qquad o \leq \cot\beta \leq \lambda$$

where $\lambda = a/h$ is the shear span ratio. The upper limit on $\cot\beta$ is imposed by the geometry of the beam. The lower limit and the bound on α ensure that the stirrups, respectively the longitudinal bars, are yielding in tension.

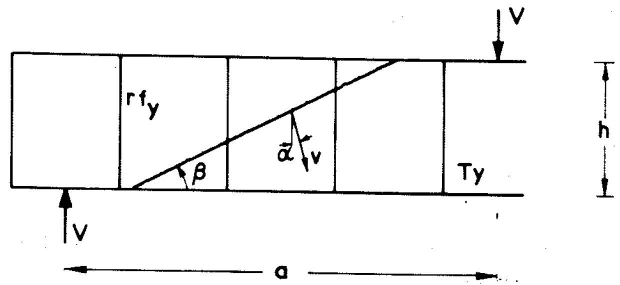


Fig. 2.1 : Failure mechanism of beam with vertical stirrups subjected to point loading

The rate of external work done by the loading is

(2.3) $W_E = vV\cos\alpha$

and the work equation $W_E = W_I$ yields the upper bound solution:

$$\tau/f_c = \frac{2\psi\cos\alpha\cos\beta + \nu[1 - \cos(\alpha-\beta)] + 2\Phi\sin\alpha\sin\beta}{2\sin\beta\cos\alpha}$$

Here $\tau = V/bh$ is the ultimate shear stress, $\nu = f_c^*/f_c$ is the concrete effectiveness factor, and we have defined the mechanical degrees of shear and main reinforcement, respectively:

(2.4) $\psi = \frac{rf_y}{f_c}$ and $\Phi = \frac{T_y}{bhf_c}$

Introducing the parameters:

(2.5) $\rho = \frac{\nu-2\psi}{\nu}$ and $\mu = \frac{\nu-2\Phi}{\nu}$

we may write the upper bound solution in the more convenient form:

(2.6) $$\tau/f_c = \nu\,\frac{1-\rho\cos\alpha\cos\beta-\mu\sin\alpha\sin\beta}{2\cos\alpha\sin\beta}$$

The lowest upper bound is found by minimizing equation (2.6) with respect to the variables α and β. Necessary conditions are:

(2.7)
$$\sin\alpha - \mu\sin\beta = 0 \qquad (\partial\tau/\partial\alpha = 0)$$
$$-\cos\beta + \rho\cos\alpha = 0 \qquad (\partial\tau/\partial\beta = 0)$$

the solutions to these equations being:

$$\text{(2.8)} \qquad \begin{aligned} \cot\beta &= \rho\sqrt{\frac{1-\mu^2}{1-\rho^2}} = \frac{\nu-2\psi}{\nu}\sqrt{\frac{\Phi(\nu-\Phi)}{\psi(\nu-\psi)}} \\ \tan\alpha &= \mu\sqrt{\frac{1-\rho^2}{1-\mu^2}} = \frac{\nu-2\Phi}{\nu}\sqrt{\frac{\psi(\nu-\psi)}{\Phi(\nu-\Phi)}} \end{aligned}$$

Inserting equations (2.7) into equation (2.6), we find the lowest upper bound:

$$\tau/f_c = \tfrac{1}{2}\nu\rho(1-\mu^2)\tan\beta \qquad \text{or by } (2.8)_1:$$

$$(2.8)_3 \qquad \tau/f_c = \frac{\nu}{2}\sqrt{(1-\mu^2)(1-\rho^2)} = \frac{2}{\nu}\sqrt{\Phi(\nu-\Phi)\psi(\nu-\psi)}$$

The condition $\alpha \geq o$ imposes a lower limit $\mu \geq \mu_1$ on the parameter μ, i.e. an upper limit $\Phi \leq \Phi_1$ on the longitudinal reinforcement degree. From equation $(2.8)_2$, we find $\mu_1=o$ and $\Phi_1=\nu/2$. For $\Phi \geq \nu/2$, the lowest upper bound is obtained with $\alpha=o$. Equation $(2.7)_2$ then yields $\cos\beta=\rho$, and by equation (2.6), the solution reduces to:

$$\text{(2.9)} \qquad \begin{aligned} \cot\beta &= \frac{\rho}{\sqrt{1-\rho^2}} = \frac{\nu-2\psi}{2\sqrt{\psi(\nu-\psi)}} \\ \tan\alpha &= o \\ \tau/f_c &= \frac{\nu}{2}\sqrt{1-\rho^2} = \sqrt{\psi(\nu-\psi)} \end{aligned}$$

This is the usual solution, corresponding to no yielding of the longitudinal reinforcement, cf. [3] p. 72. Equations (2.9) are also found from equations (2.8) by inserting $\mu=\mu_1=o$.

The requirement $\cot\beta \geq o$ implies $\rho \geq \rho_1$ or $\psi \leq \psi_1$. Equation $(2.8)_1$ shows that $\rho_1=o$ and $\psi_1=\nu/2$. For $\psi \geq \nu/2$, the lowest upper bound is obtained with $\beta=\pi/2$. Equation $(2.7)_,$ then gives $\sin\alpha=\mu$, and using equation (2.6), we find:

$$\text{(2.10)} \qquad \begin{aligned} \cot\beta &= o \\ \tan\alpha &= \frac{\mu}{\sqrt{1-\mu^2}} = \frac{\nu-2\Phi}{2\sqrt{\Phi(\nu-\Phi)}} \\ \tau/f_c &= \frac{\nu}{2}\sqrt{1-\mu^2} = \sqrt{\Phi(\nu-\Phi)} \end{aligned}$$

The same result is obtained by inserting $\rho=\rho_1=o$ into equations (2.8). This solution covers the case when the stirrups are not yielding. For $\Phi \geq \Phi_1=\nu/2$, i.e. no yielding of the longitudinal reinforcement either, the solution reduces to:

$$\cot\beta = o$$

(2.11) $$\tan\alpha = o$$

$$\tau/f_c = \nu/2$$

Cf. reference [3] , p. 72.

The limit $\cot\beta \le \lambda$ leads to the condition $\rho \le \rho_o$, or $\psi \ge \psi_o$. By equation $(2.8)_1$, we have:

$$\rho_o = \frac{\lambda}{\sqrt{\lambda^2+1-\mu^2}} \quad \text{or}$$

(2.12)

$$\psi_o = \frac{\nu}{2}\frac{\sqrt{\lambda^2+1-\mu^2}-\lambda}{\sqrt{\lambda^2+1-\mu^2}}$$

For $\psi \le \psi_o$, the lowest upper bound is obtained with $\cot\beta=\lambda$. Determining α from equation $(2.7)_,$ and inserting into equation (2.6), we find the solution:

$$\cot\beta = \lambda$$

(2.13) $$\tan\alpha = \frac{\mu}{\sqrt{1+\lambda^2-\mu^2}}$$

$$\tau/f_c = \frac{\nu}{2}\left(\sqrt{\lambda^2+1-\mu^2}-\rho\lambda\right)$$

When the main reinforcement is not yielding ($\alpha=o$), we insert $\mu=\mu_1=o$, and the solution reduces to:

$$\cot\beta = \lambda$$

(2.14) $$\tan\alpha = o$$

$$\tau/f_c = \frac{\nu}{2}\left(\sqrt{\lambda^2+1}-\rho\lambda\right)$$

These equations are valid for $\mu \le \mu_1 = o$ and $\rho \ge \rho_o$, where

$$\rho_o = \frac{\lambda}{\sqrt{\lambda^2+1}} \quad \text{or} \quad \psi_o = \frac{\nu}{2}\frac{\sqrt{\lambda^2+1}-\lambda}{\sqrt{\lambda^2+1}}$$

This result was given in reference [3] , p. 72.

2.2 Discussion

When the main reinforcement is sufficiently strong ($\Phi \geq \nu/2$), the analysis predicts $\alpha = o$, i.e. the tensile stringer is not yielding at failure of the beam. Figure 2.2a shows the failure mechanism with a deforming zone consisting of a distribution of yield lines at the inclination β, rather than the single yield line of Figure 2.1. The rate of internal work, and hence the upper bound, remains the same. Such a failure mechanism is often observed in reality, as demonstrated by Figure 2.2b, showing a beam tested at the Structural Research Laboratory (BRAESTRUP & al [8]).

The formulae derived in the preceding subsection are upper bound solutions, which means that the estimates for the ultimate shear stress τ are greater than or equal to the theoretical load-carrying capacity. Corresponding lower bound solutions are determined by the construction of statically admissible stress distributions.

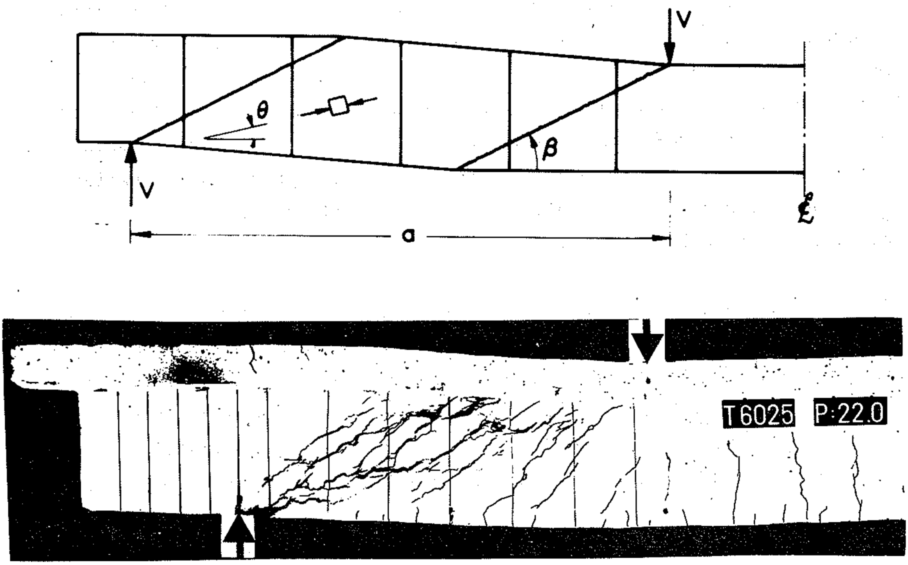


Fig. 2.2 : Shear failure of beam
a) Failure mechanism with distributed deformations
b) Test beam after failure

Beams with strong main reinforcement were treated in reference [3], and it was shown that equation $(2.9)_3$ is indeed a complete solution. The identical lower bound may be written:

$$V = bhrf_y\cot\theta \quad \text{or}$$

(2.16)

$$\tau/f_c = \psi\cot\theta$$

where $\theta=\beta/2$. The angle θ is the inclination of the concrete compression (strut inclination) of a diagonal compression field, cf. Figure 2.2a. Shear is transferred from the web to the tension stringer at the bottom of the beam and to the compression stringer at the top. The stress field is modified near the point load and the support.

The value of the strut inclination $\theta=\beta/2$ is found from equation (2.5) :

(2.17) $$\cot\theta = \sqrt{\frac{1+\rho}{1-\rho}}$$

The solution has been compared with experiments reported in the literature (cf. references [1], [6]). Figure 2.3 shows 198 results of shear tests on T-beams. As beam depth h is used the internal moment lever arm, represented by the distance from the centroid of the main reinforcement to the centre of the flange. The ultimate shear stress τ/f_c is plotted as a function of the shear reinforcement degree ψ, and compared with the theoretical prediction, equation $(2.9)_3$. The agreement is acceptable, the best value of the empirical effectiveness factor being $\nu= 0.74$.

For beams where also the shear reinforcement is sufficiently strong ($\psi\geq\nu/2$), equations (2.11) give $\beta=\pi/2$ ($\theta=\pi/4$), i.e. the stirrups are not yielding at failure of the beam. The corresponding shear strength $\tau/f_c=\nu/2$ is plotted on Figure 2.3 as the horizontal line, tangential to the circle representing equation $(2.9)_3$. The validity of this solution has been substantiated by tests on T-beams with very high shear reinforcement degrees (cf. reference [8]).

For beams without web reinforcement (ψ=o), the upper bound for the ultimate shear stress is given by $(2.12)_3$ with ρ=1. The failure mechanism of Figure 2.2a degenerates to a single yield line running from the load to the support, cf. Figure 2.4a. Note that the shear span a is defined as the clearance between the load and support platens. On Figure 2.4b is sketched a stress distribution, consisting of a single concrete strut between load and support. The shaded regions are in a state of biaxial hydrostatic compression, and it is assumed that the anchorage and the support are able to transfer the concrete forces to the tension stringer. As shown in reference [1], the lower bound corresponding to this stress field is identical with the upper bound. Thus also in this case we are dealing with the complete solution, cf. also NIELSEN & BRAESTRUP [9].

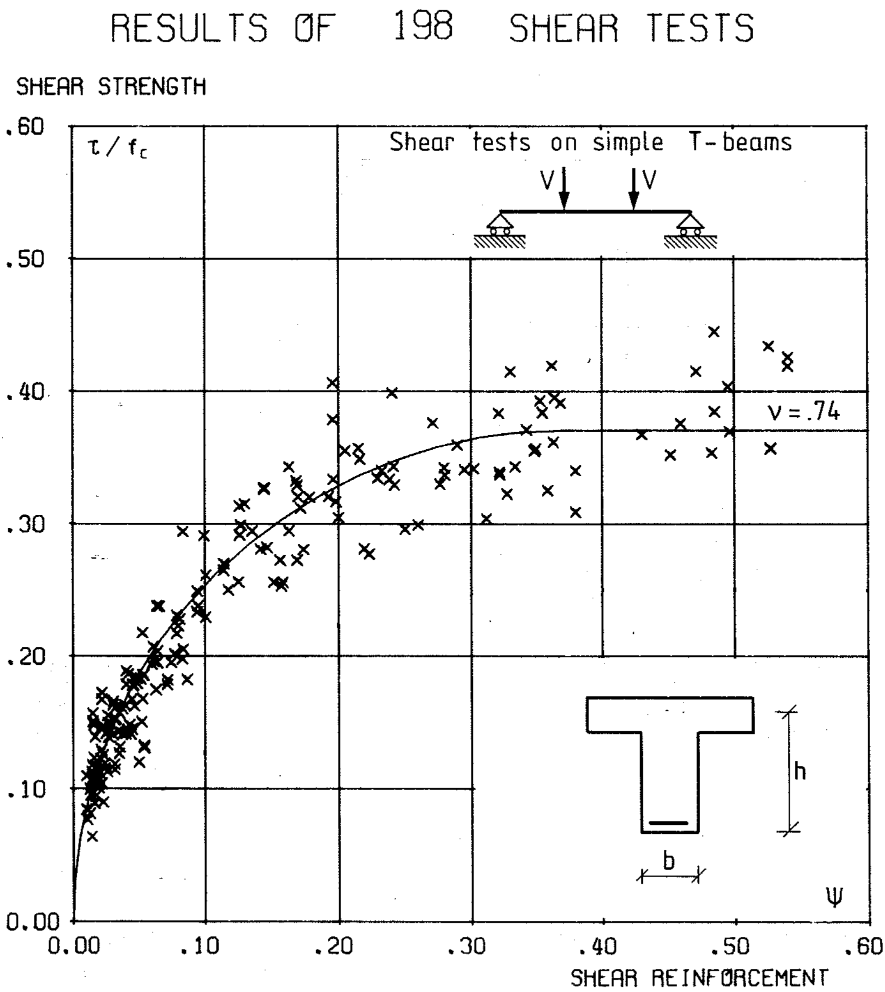


Fig. 2.3 : Results of shear tests on beams with vertical stirrups compared with theoretical prediction

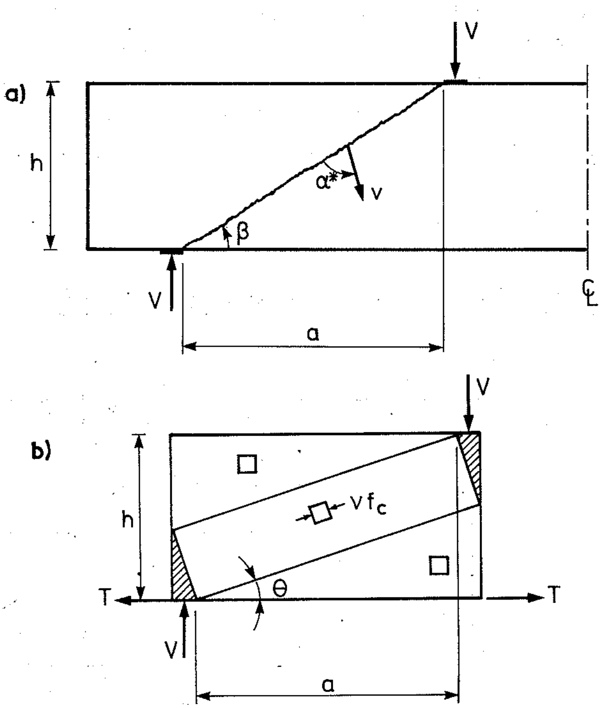


Fig. 2.4 : Shear in beams without stirrup reinforcement
a) Failure mechanism
b) Stress distribution

The analysis does not make use of the compression stringer, which is yielding in tension. This means that any top steel has to be taken into account when the longitudinal reinforcement degree Φ is calculated. We would expect the solution to be most representative of beams without compression flange, and indeed the predictions are in excellent agreement with the observed strengths of rectangular beams without stirrups. Figure 2.5 shows the results of some tests carried out by ROIKJAER [10].* As sheer depth h is used the total beam depth, and the shear strength τ/f_c is plotted as a function of the shear span ratio a/h, the reinforcement degree being constant. The points fit the theoretical curve excellently, when we assume an effectiveness factor of ν=0.46. Since ν is a

* cf. also reference (9)

measure of concrete ductility, it may be expected to decrease with increasing strength level, and this has been confirmed by analysis of a great number of test results [10]. Thus the rather low value of Figure 2.5 reflects the fact that the concrete of this series was very strong ($f_c \approx 55$MPa).

In the presence of stirrups ($\rho \neq 1$), it can be shown that equation $(2.14)_3$ (for $\Phi \leq \nu/2$) and $(2.14)_3$ (for $\Phi \geq \nu/2$) is also a lower bound (cf. JENSEN & al [11]).

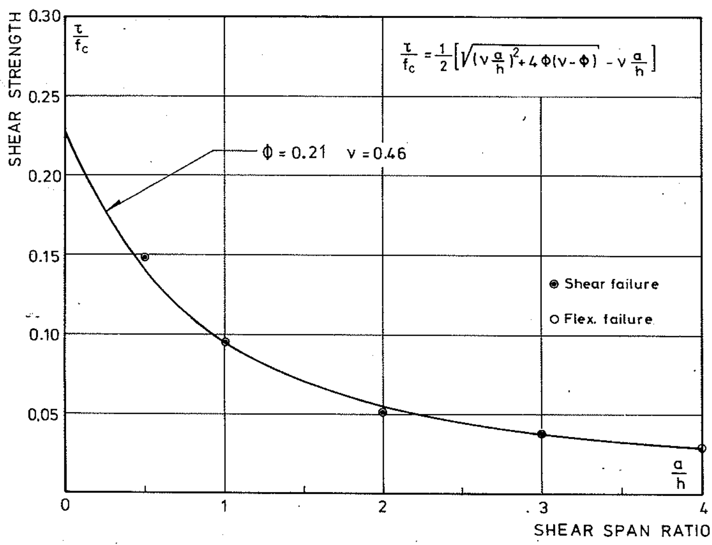


Fig. 2.5 : Results of shear tests on beams without stirrups compared with theoretical prediction

When the longitudinal reinforcement is not strong enough to ensure α=o (i.e. for $\Phi \le \nu/2$), then the shear resistance is estimated by equations $(2.13)_3$, $(2.8)_3$, or $(2.10)_3$. These upper bounds are not backed by any lower bound solutions. Equation $(2.13)_3$ applies for $\psi < \psi_o$, ψ_o being given by equation $(2.12)_2$, i.e. for beams with rather weak shear reinforcement. The requirement of most building codes, prescribing a certain minimum stirrup reinforcement, make the majority of practical beams fall outside this category. An exception is formed by deep beams. Equations $(2.8)_3$ and $(2.10)_3$ predict a reduction of the shear strength by the factor $\frac{2}{\nu}\sqrt{\Phi(\nu-\Phi)}$. For reinforcement degrees just below $\nu/2$, this factor remains close to unity (for $\Phi=\nu/3$, we get 0.94). At any rate, most beams failing in shear will have high longitudinal reinforcement degrees, in order not to fail in flexure. The 198 beams, plotted on Figure 2.3 to test equation $(2.9)_3$, all have $\Phi \ge 0.3$. Thus for reasonably designed beams, equations $(2.8)_3$ and $(2.10)_3$ are not very valuable. Their importance arises in connection with axial force, cf. reference [1]. The equations may become topical, however, in the case of bond or anchorage failure, because this reduces the tensile stringer strength available in the shear mechanism without affecting the flexural capacity. An anchorage defect may be almost impossible to distinguish from a proper shear failure, and it cannot be excluded that a number of the failures reported in the literature as due to shear are actually governed by imperfect anchorage.

The various solutions are summarized in Section 6, where also the corresponding domains are visualized on Figure 6.1.

3. CONCENTRATED LOADING, INCLINED STIRRUPS

3.1 Upper bound solutions

We now consider shear failure of a beam with stirrups inclined at the angle γ to the beam axis (cf. Figure 3.1). The rate of internal work is modified to:

$$W_I = rf_y \sin(\beta+\gamma) \frac{bh}{\sin\beta} v\sin(\alpha+\gamma) + \tfrac{1}{2} v f_c^* \left[1-\cos(\beta-\alpha)\right] \frac{bh}{\sin\beta} + T_y v\sin\alpha \tag{3.1}$$

Owing to the stirrup inclination, yield line inclinations $\beta>\pi/2$ may be permitted without causing compression of the stirrups. Thus the ranges of the variables α and β are:

$$\alpha \geq o \qquad \text{and} \qquad -\cot\gamma \leq \cot\beta \leq \lambda \tag{3.2}$$

The rate of external work is still given by equation (2.3), and the work equation yields the upper bound solution:

$$\tau/f_c = \frac{2\psi\sin(\beta+\gamma)\sin(\alpha+\gamma) + \nu\left[1-\cos(\beta-\alpha)\right] + 2\Phi\sin\alpha\sin\beta}{2\sin\beta\cos\alpha}$$

where the reinforcement degrees ψ and Φ are defined by equations (2.4).

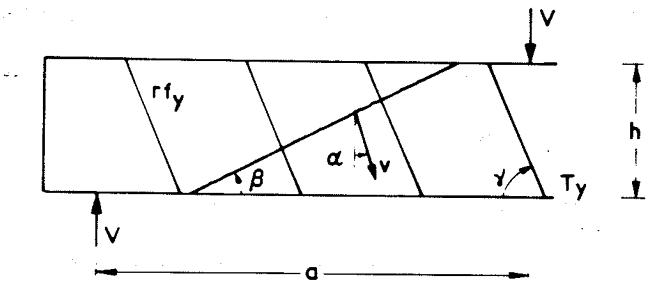


Fig. 3.1 : Failure mechanism of beam with inclined stirrups subjected to point loading

We introduce the parameters:

(3.3)
$$\rho = 1-\frac{2\psi}{\nu}\sin^2\gamma \qquad \mu = 1-\frac{2\Phi}{\nu} - \frac{2\psi}{\nu}\cos^2\gamma$$
$$\bar{\mu} = 1-\frac{2\Phi}{\nu} \qquad \kappa = \frac{2\psi}{\nu}\cos\gamma\sin\gamma$$

In the case of vertical stirrups ($\gamma=\pi/2$), we get $\kappa=o$ and the definitions of ρ and $\mu=\bar{\mu}$ reduce to equations (2.5). The upper bound solution may now be expressed in the form:

(3.4)
$$\tau/f_c = \nu\,\frac{1-\rho\cos\alpha\cos\beta-\mu\sin\alpha\sin\beta+\kappa\sin\alpha\cos\beta+\kappa\cos\alpha\sin\beta}{2\sin\beta\cos\alpha}$$

Minimization with respect to the variables α and β yields the equations:

(3.5)
$$\sin\alpha-\mu\sin\beta+\kappa\cos\beta = o \qquad (\partial\tau/\partial\alpha = o)$$
$$\rho\cos\alpha-\cos\beta-\kappa\sin\alpha = o \qquad (\partial\tau/\partial\beta = o)$$

The solutions to these equations are:

(3.6)
$$\cot\beta = \frac{1}{1-\kappa^2}\left[\rho\sqrt{\frac{1-\mu^2-\kappa^2}{1-\rho^2-\kappa^2}} - \kappa\mu\right]$$
$$\tan\alpha = \frac{1}{1-\kappa^2}\left[\mu\sqrt{\frac{1-\rho^2-\kappa^2}{1-\mu^2-\kappa^2}} - \mu\rho\right]$$

Inserting equations (3.5) into equation (3.4), we find the lowest upper bound:

$$\tau/f_c = \frac{\nu}{2\rho}\left[(1-\rho^2-\kappa^2)\cot\beta+\kappa(\rho+\mu)\right]$$

Using equation $(3.6)_1$, this becomes:

$(3.6)_3$
$$\tau/f_c = \frac{\nu}{2(1-\kappa^2)}\left[\sqrt{1-\rho^2-\kappa^2}\sqrt{1-\mu^2-\kappa^2} + \kappa(1-\kappa^2+\mu\rho)\right]$$

For vertical stirrups ($\kappa=o$), equations (3.6) reduce to equations (2.8).

The lower limit $\mu=\mu_1$ on the parameter μ, imposed by the condition $\alpha\geq o$, is found from equation $(3.6)_2$:

$(3.7)_1$
$$\mu_1 = \frac{\kappa\rho}{\sqrt{1-\rho^2}}$$

This is equivalent to an upper limit $\Phi \leq \Phi_1$ on the longitudinal reinforcement degree. Using equations (3.3), we get:

$$(3.7)_2 \qquad \Phi_1 = \frac{\nu}{2} - \psi\cos\gamma\left(\cos\gamma + \frac{\nu - 2\psi\sin^2\gamma}{2\sqrt{\psi(\nu-\psi\sin^2\gamma)}}\right)$$

For $\Phi \geq \Phi_1$, the lowest upper bound is obtained with α=o. Equation $(3.5)_2$ then yields $\cos\beta=\rho$, and using equation (3.4), we find:

$$(3.8) \qquad \begin{aligned} \cot\beta &= \frac{\rho}{\sqrt{1-\rho^2}} = \frac{\nu-2\psi\sin^2\gamma}{2\sqrt{\psi\sin^2\gamma(\nu-\psi\sin^2\gamma}} \\ \tan\alpha &= o \\ \tau/f_c &= \frac{\nu}{2}\left(\sqrt{1-\rho^2}+\kappa\right) = \sqrt{\psi\sin^2\gamma(\nu-\psi\sin^2\gamma)} + \psi\cos\gamma\sin\gamma \end{aligned}$$

The same result is obtained from equation (3.6) by inserting $\mu=\mu_1$, given by equation (3.7). This is the solution corresponding to no yielding of the main reinforcement (cf. reference [3] p. 84). In the case of vertical stirrups ($\gamma=\pi/2$), the solution reduces to equations (2.9).

Equation $(3.6)_1$ determines the lower limit $\rho=\rho_1$ on the parameter ρ, implied by the condition $\cot\beta \geq -\cot\gamma$:

$$(3.9)_1 \qquad \rho_1 = -\frac{\cot\gamma-\kappa(\mu+\kappa\cot\gamma)}{\sqrt{1+\cot^2\gamma-(\mu+\kappa\cot\gamma)^2}}$$

Inserting equations (3.3), we find the equivalent upper limit $\psi=\psi_1$ on the shear reinforcement degree:

$$(3.9)_2 \qquad \psi_1 = \frac{\nu}{2}\left(1+\cot\gamma\,\frac{\cot\gamma+\sqrt{1+\cot^2\gamma-\bar{\mu}}}{1+\bar{\mu}}\right)^2$$

Here we have introduced $\bar{\mu}=\mu+\kappa\cot\gamma$, defined by equation $(3.3)_4$. For vertical stirrups ($\gamma=\pi/2$),this reduced to ρ_1=o and $\psi_1=\nu/2$, as found in Section 2.1.

When the main reinforcement is not yielding, we solve equations $(3.7)_1$ and $(3.9)_1$, and find:

$$(3.10) \qquad \mu_1 = -\kappa\cot\gamma \qquad \text{or} \qquad \Phi_1 = \nu/2$$

$$(3.11) \qquad \rho_1 = -\cos\gamma \qquad \text{or} \qquad \psi_1 = \frac{\nu}{2}\,\frac{1+\cos\gamma}{\sin^2\gamma}$$

These limits were given in reference [3], p. 84.

For $\psi \geq \psi_1$, the lowest upper bound is obtained with $\cot\beta = -\cot\gamma$. Determining α from equation $(3.5)_1$ and inserting the values of α and β into equation (3.4), we find the solution:

$$\cot\beta = -\cot\gamma$$

$$(3.12) \qquad \tan\gamma = \frac{\bar{\mu}}{\sqrt{1+\cot^2\gamma-\bar{\mu}^2}}$$

$$\tau/f_c = \frac{\nu}{2}\left(\sqrt{1+\cot^2\gamma-\bar{\mu}^2}+\cot\gamma\right)$$

The same result is obtained by putting $\rho=\rho_1$ in equations (3.6), and using equations (3.3). For vertical stirrups ($\gamma=\pi/2$), the solution reduces to equations (2.10).

From equation $(3.12)_2$, we see that no yielding of the main reinforcement corresponds to $\bar{\mu}=o$, as found above. Inserting into equations (3.12), we find:

$$\cot\beta = -\cot\gamma$$

$$(3.13) \qquad \tan\alpha = o$$

$$\tau/f_c = \frac{\nu}{2}\cot\frac{\gamma}{2}$$

This solution was derived in reference [3], p. 84.

The limit $\cot\beta \leq \lambda$ leads to the condition $\rho \leq \rho_o$, where ρ_o is determined by equation $(3.6)_1$ with $\cot\beta=\lambda$:

$$(3.14)_1 \qquad \rho_o = \frac{\lambda+\kappa(\mu-\lambda\kappa)}{\sqrt{1+\lambda^2-(\mu-\lambda\kappa)^2}}$$

The corresponding lower limit ψ_o on the shear reinforcement degree is found by inserting equations (3.3). An explicit expression is most easily obtained by solving with respect to μ:

$$(3.14)_2 \qquad \mu = \frac{-\lambda\kappa(1-\kappa^2-\rho^2) + \rho\sqrt{(1+\lambda^2)(\kappa^2+\rho^2)-\lambda^2}}{\kappa^2+\rho^2}$$

For vertical stirrups ($\kappa=o$), the limit reduces to equations (2.12).

When $\psi \leq \psi_o$, the lowest upper bound is obtained with $\cot\beta=\lambda$. The angle α is determined by equation $(3.5)_1$, and inserting into equation (3.4), we find the solution:

$$\cot\beta = \lambda$$

(3.15) $$\tan\alpha = \frac{\mu-\lambda\kappa}{\sqrt{1+\lambda^2-(\mu-\lambda\kappa)^2}}$$

$$\tau/f_c = \frac{\nu}{2}\left(\sqrt{1+\lambda^2-(\mu-\lambda\kappa)^2} - \rho\lambda+\kappa\right)$$

For vertical stirrups ($\kappa=o$), equations (3.15) reduce to equations (2.13).

When the main reinforcement is not yielding, we solve equations $(3.7)_1$ and $(3.14)_1$ to find:

(3.16) $$\mu_1 = \kappa\lambda$$

(3.17) $$\rho_o = \frac{\lambda}{\sqrt{\lambda^2+1}}$$

That $\mu=\mu_1=\kappa\lambda$ corresponds to $\alpha=o$ is also seen from equation $(3.15)_2$. The solution then reduces to:

$$\cot\beta = \lambda$$

(3.18) $$\tan\alpha = o$$

$$\tau/f_c = \frac{\nu}{2}\left(\sqrt{1+\lambda^2} - \rho\lambda+\kappa\right)$$

This result was found in reference [3], p. 84. The equations are valid for $\mu \leq \mu_1$ and $\rho \geq \rho_o$, the limits being given by equations (3.16) and (3.17). For vertical stirrups ($\kappa=o$), equations (3.18) reduce to equations (2.14).

3.2 Discussion

Beams for which the longitudinal reinforcement is sufficiently strong to prevent yielding of the tensile stringer at failure were treated in reference [3]. It was found that equation $(3.9)_3$ represents the complete solution and a corresponding stress distribution was determined. It consists of a diagonal compression field with a strut inclination $\theta=\beta/2$, i.e. half the yield line inclination.

The lower bound solution may be written:

$$V = bhrf_y \sin^2\gamma(\cot\theta + \cot\gamma) \quad \text{or}$$

(3.16)

$$\tau/f_c = \psi\sin^2\gamma(\cot\theta+\cot\gamma) = \frac{\nu}{2}\left[(1-\rho)\cot\theta+\kappa\right]$$

The corresponding strut inclination $\theta=\beta/2$ is found from equation $(3.8)_1$.

(3.17) $$\cot\theta = \sqrt{\frac{1+\rho}{1-\rho}}$$

This expression is identical with equation (2.17), but note that the definition of ρ is different.

The shear reinforcement degree ψ is defined in such a way (cf. equations (1.1) and $(2.4)_1$) that irrespective of stirrup inclination it is proportional to the amount of active web reinforcement, i.e. stirrup volume minus anchorage devices (hooks, horizontal legs, etc.). The yield line inclination is given by equation $(3.9)_1$, and we note that for the same value of ψ, the yield lines are the flatter, the more inclined the stirrups are. For $\psi=\nu/2\sin^2\gamma$, the yield lines are vertical, and for greater shear reinforcement degrees, the yield line inclination exceeds $\pi/2$ until it reaches the limit $\cot\beta=-\cot\gamma$ for $\psi=\nu(1+\cot\gamma)/2\sin^2\gamma$. In this case the yield lines are parallel to the stirrups, and the beam is overreinforced in shear.

The effect of stirrup inclination upon the shear strength is also dependent upon the amount of web reinforcement. For small values of ψ, there is a slight reduction in strength, whereas for great values we get a substantial increase. The range of applicability of equation $(3.9)_3$ is larger than that of equation $(2.9)_3$, which means that the shear resistance keeps increasing with the web reinforcement degree longer than when the stirrups are vertical. The parameter domains of the various solutions are sketched on Figure 6.2 of Section 6, which also contains a summary of the results.

The optimal stirrup inclination γ_M for a given shear reinforcement degree ψ is found ([3] p. 85) to be $\cot\gamma_M = \sqrt{\psi}$, and the corresponding shear strength is $\tau/f_c = \sqrt{\psi}$. When the stirrups are optimally inclined, they are perpendicular to the concrete compression of the lower bound solution. Equation $(3.9)_3$ is plotted on Figure 3.2 for $\gamma=\pi/2$, $\gamma=\pi/4$, and $\gamma=\gamma_M$. In all cases the effectiveness factor is inserted as $\nu=1.0$.

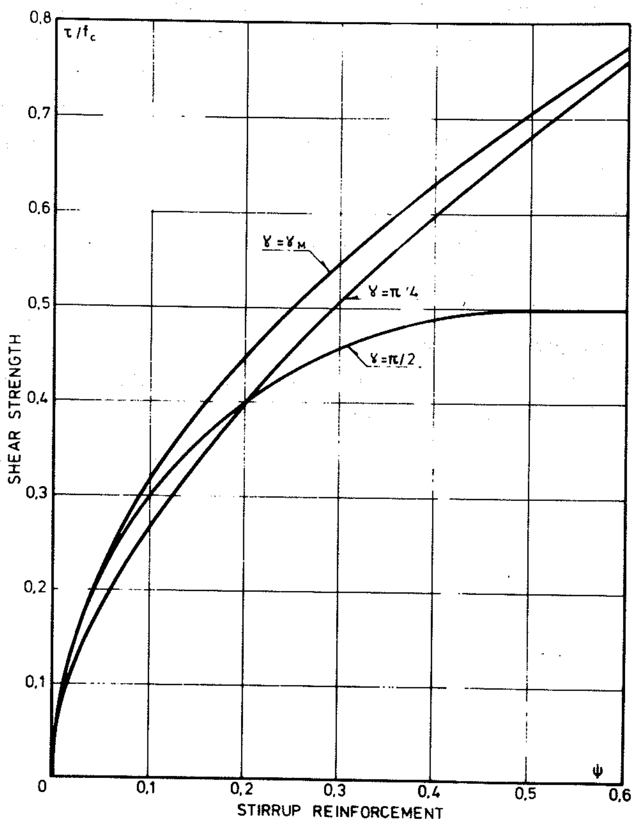


Fig. 3.2 : Effect of stirrup inclination on shear strength of beams subjected to point loading

Experimental evidence on beams with inclined stirrups is not so plentiful as for vertical stirrups. A few tests have been carried out at the Structural Research Laboratory (cf. Reference [12]). They show that equation $(3.9)_3$ may be applied, and that the effectiveness factor ν can be taken to be the same as for vertical stirrups. Tests on thin-webbed T-beams with very high shear reinforcement degrees (e.g. ROBINSON & DEMORIEUX [13]) have demonstrated that a suitable inclination of the stirrups leads to a great increase in shear resistance, as predicted by the theoretical solution.

As yet, no lower bound solution corresponding to equation $(3.18)_3$, valid for low shear reinforcement degrees, has been found. The critical value ψ_0 is slightly greater than for vertical stirrups, but still the requirements of minimum shear reinforcement will exclude most cases with $\psi<\psi_0$. This is even more evident when the main reinforcement is yielding, in which case ψ_0 is determined implicitly by equations (3.14) (cf. Figure 6.2).

When the main reinforcement is too weak to ensure a vertical deformation rate at failure, the shear strengths, as given by equations $(3.6)_3$ and $(3.12)_3$, are reduced somewhat. As for beams with vertical stirrups, this reduction is not likely to be of much practical interest, since a substantial stringer strength is necessary to prevent flexural failure. This is especially true for the large web reinforcement degrees that imply high shear strength.

4. DISTRIBUTED LOADING, VERTICAL STIRRUPS

4.1 Upper bound solutions

Consider a beam subjected to the uniformly distributed load pb over the shear span. On Figure 4.1 is sketched a shear failure mechanism consisting of a yield line starting at the support.

The rate W_I of internal work is given by equation (2.1), where the variables α and β are subject to the limits (2.2). The rate of external work is:

(4.1) $$W_E = vpb(a-h\cot\beta)\cos\alpha$$

The work equation $W_E=W_I$ yields the upper bound solution:

(4.2) $$\tau/f_c = \nu\lambda \frac{1-\rho\cos\alpha\cos\beta-\mu\sin\alpha\sin\beta}{2(\lambda\sin\beta-\cos\beta)\cos\alpha}$$

Here $\tau=pba/bh=\lambda p$ is the ultimate shear stress at the support section, and the parameters ρ and μ are given by equations (2.5), the reinforcement degrees ψ and Φ being defined by equations (2.4).

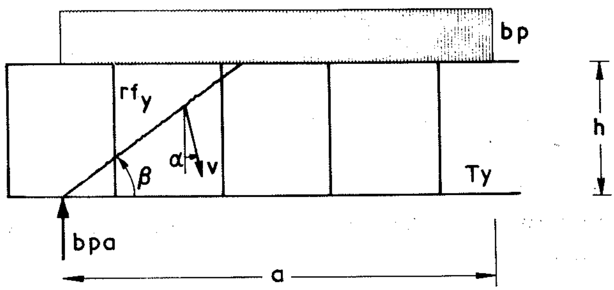


Fig. 4.1 : Failure mechanism of beam with vertical stirrups subjected to distributed loading

The minimum upper bound is obtained for values of α and β satisfying the equations:

$$\text{(4.3)} \quad \begin{aligned} \sin\alpha-\mu\sin\beta &= o & (\partial\tau/\partial\alpha=o) \\ \lambda\rho\cos\alpha+\mu\sin\alpha-\lambda\cos\beta-\sin\beta &= o & (\partial\tau/\partial\beta=o) \end{aligned}$$

The solutions to equations (4.3) are:

$$\text{(4.4)} \quad \begin{aligned} \cot\beta &= \frac{1-\mu^2}{\lambda(1-\rho^2)}\left[\rho\sqrt{1+\lambda^2\frac{1-\rho^2}{1-\mu^2}} - 1\right] \\ \tan\alpha &= \frac{\lambda\mu}{\lambda^2+1-\mu^2}\left[\rho + \sqrt{1+\lambda^2\frac{1-\rho^2}{1-\mu^2}}\right] \end{aligned}$$

Inserting equations (4.3) into equation (4.2), we find the lowest upper bound:

$$\tau/f_c = \frac{\nu}{2}\,\frac{(1-\mu^2)+\lambda(1-\rho^2)\cot\beta}{\lambda\rho\ -\rho\cot\beta} \quad \text{, or by } (4.4)_1\text{:}$$

$$(4.4)_3 \quad \tau/f_c = \frac{\nu\lambda(1-\mu^2)}{2(\lambda^2+1-\mu^2)}\left[\rho + \sqrt{1+\lambda^2\,\frac{1-\rho^2}{1-\mu^2}}\right]$$

The condition $\alpha\geq o$ imposes the limit $\mu\geq\mu_1=o$ or $\Phi\leq\Phi_1=\nu/2$. For $\Phi\geq\nu/2$, the lowest upper bound is obtained with $\alpha=o$. Determining β from equation $(4.3)_2$ and inserting into equation (4.2), we find:

$$\text{(4.5)} \quad \begin{aligned} \cot\beta &= \frac{1}{\lambda(1-\rho^2)}\left(\rho\sqrt{1+\lambda^2(1-\rho^2)} - 1\right) \\ \tan\alpha &= o \\ \tau/f_c &= \frac{\nu\lambda}{2(1+\lambda^2)}\left(\rho + \sqrt{1+\lambda^2(1-\rho^2)}\right) \end{aligned}$$

The same result is obtained by putting $\mu=\mu_1=o$ in equations (4.4). This is the solution corresponding to no yielding of the longitudinal reinforcement, derived in reference [3], p. 80.

The condition $\cot\beta\geq o$ requires $\rho\geq\rho_1$ or $\psi\leq\psi_1$. From equation $(4.4)_1$, we find:

$$\text{(4.6)} \quad \rho_1 = \frac{\sqrt{1-\mu^2}}{\lambda} \quad \text{or} \quad \psi_1 = \frac{\nu}{2}\,\frac{\lambda-\sqrt{1-\mu^2}}{\lambda}$$

For $\psi\geq\psi_1$, the lowest upper bound is obtained with $\beta=\pi/2$. Then, by equations $(4.3)_1$ and (4.2), the solution is:

(4.7)
$$\cot\beta = o$$
$$\tan\alpha = \frac{\mu}{\sqrt{1-\mu^2}}$$
$$\tau/f_c = \frac{\nu}{2}\sqrt{1-\mu^2}$$

This is identical with equations (2.10). The same result is obtained by inserting $\rho=\rho_1$ into equations (4.4).

When also the main reinforcement is very strong, i.e. for $\Phi\geq\Phi_1=\nu/2$, the solution reduces to:

(4.8)
$$\cot\beta = o$$
$$\tan\alpha = o$$
$$\tau/f_c = \nu/2$$

This is identical with equations (2.11). The solution is valid for $\rho\leq\rho_1$, found by putting $\mu=\mu_1=o$ in equations (4.6):

(4.9) $$\rho_1 = \frac{1}{\lambda} \qquad \text{or} \qquad \psi_1 = \frac{\nu}{2}\,\frac{\lambda-1}{\lambda}$$

cf. reference [3], p. 80. Equations (4.8) are also found by inserting $\rho=\rho_1=1/\lambda$ into equations (4.5).

The condition $\cot\beta\leq\lambda$ is always satisfied, even for very small shear reinforcement degrees. Indeed, for $\rho\to1$, equation $(4.4)_1$ yields:

$$\cot\beta \to \frac{\lambda^2-(1-\mu^2)}{2\lambda} < \lambda$$

For beams without stirrups, the shear strength is given by equation $(4.4)_3$ with $\rho=1$, viz:

(4.10) $$\tau/f_c = \nu\,\frac{\lambda(1-\mu^2)}{\lambda^2+1-\mu^2}$$

When the main reinforcement is sufficiently strong ($\Phi\geq\nu/2$), we insert $\mu=\mu_1=o$, and the shear strength reduces to $\tau/f_c=\nu\lambda/(\lambda^2+1)$ (cf. reference [3], p. 80). Equation (4.10) is valid as long as equation (4.6) yields a positive value of ψ_1, i.e. for $\lambda>\sqrt{1-\mu^2}$. For smaller shear span ratios, the upper bound solution for the shear strength is $\tau/f_c=\nu/2$, irrespective of the reinforcement.

4.2 Discussion

In reference [3], a stress distribution was suggested for beams with strong main reinforcement ($\Phi \geq \nu/2$). It is not as simple as for point loading. Assuming a constant strut inclination θ, the tensile stress in the stirrups and the compressive stress in the web concrete varies along the shear span, being maximum at the support. Introducing an additional assumption concerning the optimal stress state, the corresponding lower bound was shown to be identical with equation $(4.5)_3$ (respectively $(4.8)_3$), which accordingly represents the complete solution.

The lower bound solution may be written:

$$V - bph\cot\theta = bhrf_y\cot\theta \,, \qquad \text{or}$$

$$\lambda\frac{p}{f_c} - \frac{p}{f_c}\cot\theta = \psi\cot\theta \tag{4.11}$$

Equation (4.11) expresses vertical equilibrium along a concrete strut starting at the support, cf. Figure 4.2. The optimal inclination of the concrete compression is found to be $\theta = \beta/2$ (cf. also Figure 1.2b). Equation $(4.5)_1$ gives:

$$\cot\theta = \frac{\sqrt{1+\lambda^2(1-\rho^2)}-1}{\lambda(1-\rho)} \tag{4.12}$$

Comparing with equation (2.17), we note that the strut inclination (and hence also the yield line inclination) is steeper by distributed than by point loading for the same amount of shear reinforcement.

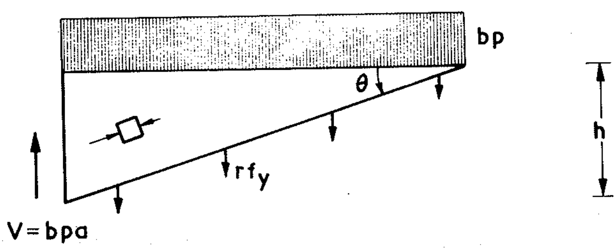


Fig. 4.2 : End zone of beam with vertical stirrups, bounded by section parallel to concrete compression

The fact that the load is distributed produces a considerable increase in shear capacity corresponding to a given shear reinforcement degree ψ. In reference [3] it was shown that the ultimate shear stress $\tau=\lambda p$ for distributed loading is obtained from the solution for point loading by replacing the parameter ψ by the quantity $\psi+p/f_c$. This result is also valid when the main reinforcement is not yielding, i.e. for $\Phi<\nu/2$. Indeed, equation $(2.8)_3$ yields:

$$\tau/f_c = \lambda p/f_c = \frac{\nu}{2}\sqrt{1-\mu^2}\sqrt{1-(\rho-\frac{2}{\nu}\frac{p}{f_c})^2}$$

with the solution:

$$(4.13) \qquad p/f_c = \frac{\nu\rho(1-\mu^2)+\nu\sqrt{(1-\mu^2)+\lambda^2(1-\mu^2)(1-\rho^2)}}{2(\lambda^2+1-\mu^2)}$$

This equation is equivalent with equation $(4.4)_3$, q.e.d. Thus in case of distributed loading, the shear reinforcement degree ψ may be reduced by the amount p/f_c, without affecting the shear strength. If the relative shear strength τ/f_c is plotted as a function of the shear reinforcement degree ψ, then the solution for distributed loading is obtained by shifting the curve for concentrated loading the distance p/f_c to the left.

When there is no yielding of the main reinforcement (the case $\alpha=o$ for $\Phi\geq\nu/2$), this result is illustrated on Figure 4.3. The solution for distributed loading, equation $(4.5)_3$, is plotted for a shear span ratio of $\lambda=3$ and compared with the curve corresponding to point loading, equation $(2.9)_3$. The effectiveness factor ν is inserted as unity.

The reduction in necessary shear reinforcement is also apparent by comparison of equations (4.11) and (2.16). Indeed, inserting equation (4.12) into equation (4.11) we arrive at equation (4.13) with $\mu=o$, i.e. equation $(4.5)_3$. Thus, when the load is distributed, the stirrup reinforcement rf_y may be designed for a shear load $V-bph\cot\Theta$, i.e. for the shear force in the distance $h\cot\Theta$ from the support (cf. references [1], [14]).

Figure 4.3 also represents equation (2.13), valid for point loading and $\psi\leq\psi_o$. No such modification for small reinforcement degrees is necessary for distributed loading. The solutions are summarized in Section 6, the domains being sketched on Figure 6.3.

Because of the high shear strength (cf. Figure 4.3), most beams with distributed loading fail in flexure, and the number of shear tests reported is quite limited. In reference [3] is cited a single result, which is in reasonable agreement with the theory.

For beams without stirrups, the shear strength is estimated by equation (4.10). For strong main reinforcement ($\Phi \geq \nu/2$), an identical lower bound is easily found, provided that shear can be transferred to a compression stringer (cf. Figure 4.2). For rectangular beams, the complete solution may be shown to be smaller (see JENSEN & al. [11]).

For $\Phi < \nu/2$, the main reinforcement is not sufficiently strong to ensure $\alpha = o$, and in that case no lower bound solutions have been found corresponding to the upper bounds. Equation $(4.4)_3$ predicts a reduction of the shear strength, this reduction being the same as for point loading, when the shear reinforcement degree is high ($\psi \geq \nu(\lambda-1)/2\lambda$), since then equation $(4.4)_3$ reduces to equation $(2.10)_3$. For weaker shear reinforcement, the reduction is somewhat higher, although still modest. For $\psi = o$ and $\lambda = 3$, it amounts to a factor of 0.90 for $\Phi = \nu/3$.

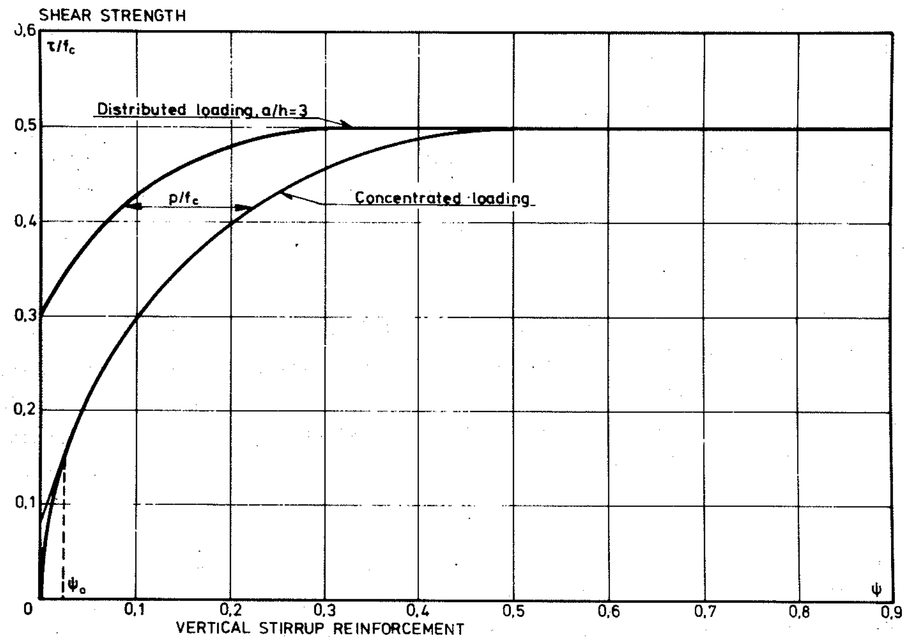


Fig. 4.3 : Comparison of shear strength of beams subjected to distributed and concentrated loading

5. DISTRIBUTED LOADING, INCLINED STIRRUPS

5.1 Upper bound solutions

We now consider a beam subjected to the distributed load bp and with stirrups inclined at the angle γ. The shear failure mechanism is sketched on Figure 5.1. The rate of internal work W_I is given by equation (3.1), the variable angles α and β being subjected to the limits (3.2) The rate of external work W_E is determined by equation (4.1). Since we may have $\cot\beta < o$, it is assumed that the beam is loaded also to the left of the support. The work equation $W_E = W_I$ yields the upper bound solution:

$$\tau/f_c = \frac{2\psi\sin(\beta+\gamma)\sin(\alpha+\gamma)+\nu[1-\cos(\beta-\alpha)]+2\Phi\sin\alpha\sin\beta}{2(\lambda-\cot\beta)\cos\alpha\sin\beta}$$

Here the reinforcement degrees ψ and Φ are defined by equations (2.4). Introducing the parameters ρ, μ, $\bar{\mu}$, and κ, defined by equations (3.3), we write the upper bound as:

$$(5.1) \qquad \tau/f_c = \nu\lambda\frac{1-\rho\cos\alpha\cos\beta-\mu\sin\alpha\sin\beta+\kappa\sin\alpha\cos\beta+\kappa\cos\alpha\sin\beta}{2(\lambda\sin\beta-\cos\beta)\cos\alpha}$$

Minimization with respect to the variables α and β, yields the equations:

$$(5.2) \qquad \begin{aligned} &\sin\alpha-\mu\sin\beta+\kappa\cos\beta = o && (\partial\tau/\partial\alpha = o) \\ &\lambda\rho\cos\alpha+\mu\sin\alpha-\kappa(\lambda\sin\alpha+\cos\alpha)-\lambda\cos\beta-\sin\beta = o && (\partial\tau/\partial\beta = o) \end{aligned}$$

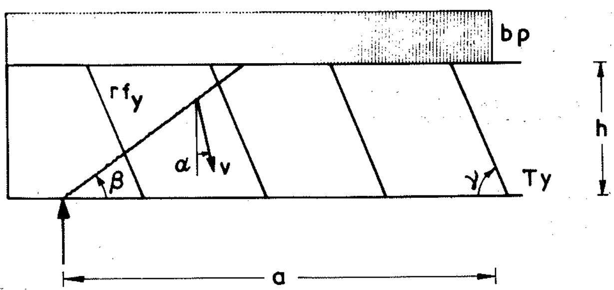


Fig. 5.1 : Failure mechanism of beam with inclined stirrups subjected to distributed loading

These equations can be solved with respect to α and β. If we introduce the additional parameter

$$(5.3) \qquad \eta = \frac{\lambda(1-\kappa^2-\rho^2)+2\kappa(\mu+\rho)}{\lambda(1-\kappa^2-\mu^2)} \quad ,$$

the solutions may be written:

$$(5.4) \qquad \cot\beta = \frac{(\lambda\rho-\kappa)\sqrt{1+\lambda^2\eta}-(\lambda+(1+\lambda^2\eta)\kappa\mu)}{(1-\kappa^2)\lambda^2\eta-\kappa^2}$$

$$\tan\alpha = \frac{(\lambda\mu+\kappa)\sqrt{1+\lambda^2\eta}-(\kappa\lambda-\mu)(\lambda\rho-\kappa)}{1+\lambda^2-(\kappa\lambda-\mu)^2}$$

For vertical stirrups ($\kappa=o$), these equations reduce to equations $(4.4)_1$ and $(4.4)_2$.

Inserting equations (5.2) into equation (5.1), we find the lowest upper bound:

$$\tau/f_c = \frac{\nu\lambda}{2}\,\frac{1-\mu^2-\kappa^2+\lambda\kappa(\rho+\mu)+[\lambda(1-\kappa^2-\rho^2)+\kappa(\rho+\mu)]\cot\beta}{(\lambda\rho-\kappa)(\lambda-\cot\beta)}$$

Using equation $(5.4)_1$ and performing some tedious manipulations, we may write this equation:

$$(5.4)_3 \qquad \tau/f_c = \frac{\nu\lambda}{2}\,\frac{\lambda\kappa+\rho-(\mu-\lambda\kappa)(\mu\rho-\kappa^2)+(1-\kappa^2-\mu^2)\sqrt{1+\lambda^2\eta}}{1+\lambda^2-(\mu-\lambda\kappa)^2}$$

For vertical stirrups ($\kappa=o$), the upper bound solution reduces to equation $(4.4)_3$.

The requirement that $\alpha\geq o$ imposes the condition $\mu\geq\mu_1$. The value of μ_1 is found from equation $(5.4)_2$. The solution is:

$$(5.5) \qquad \mu_1 = \kappa\,\frac{(\lambda\rho-\kappa)\sqrt{1+\lambda^2-(\lambda\rho-\kappa)^2}-\lambda}{\lambda^2-(\lambda\rho-\kappa)^2}$$

corresponding to $\lambda^2\eta=\lambda^2-(\lambda\rho-\kappa)^2$. The corresponding value of $\phi=\phi_1(\psi)$ is found by inserting equations (3.3).

For $\mu<\mu_1$, the lowest upper bound is obtained with $\alpha=o$. Determining β from equation $(5.2)_2$, and inserting into equation (5.1), we find the solution:

$$\cot\beta = \frac{(\lambda\rho-\kappa)\sqrt{1+\lambda^2-(\lambda\rho-\kappa)^2}-\lambda}{\lambda^2-(\lambda\rho-\kappa)^2}$$

(5.6) $$\tan\alpha = o$$

$$\tau/f_c = \frac{\nu\lambda}{2}\,\frac{\kappa\lambda+\rho+\sqrt{1+\lambda^2-(\lambda\rho-\kappa)^2}}{1+\lambda^2}$$

Note that in this case $\mu_1=\kappa\cot\beta$, as for point loading, cf. equations $(3.7)_1$ and $(3.8)_1$. For vertical stirrups (κ=o), equations (5.6) reduce to equations (4.5).

The condition $\cot\beta\geq-\cot\gamma$ requires $\rho\geq\rho_1$. From equation $(5.4)_1$, we find after some manipulation:

$(5.7)_1$ $$\lambda\rho_1-\kappa = \frac{1-\lambda\cot\gamma-(\mu-\lambda\kappa)(\mu+\kappa\cot\gamma)}{\sqrt{1+\cot^2\gamma-(\mu+\kappa\cot\gamma)^2}}$$

Inserting $\rho=1-\kappa\tan\gamma$ and $\mu=\bar{\mu}-\kappa\cot\gamma$, and solving with respect to κ, we find:

$(5.7)_2$ $$\psi_1 = \frac{\nu}{2}\;\frac{(\lambda+\bar{\mu}\cot\gamma)(1+\bar{\mu})+\lambda\cot^2\gamma-(1+\bar{\mu}-\lambda\cot\gamma)\sqrt{1+\cot^2\gamma-\bar{\mu}^2}}{(1+\bar{\mu})(\lambda+\cot\gamma)}$$

For vertical stirrups ($\gamma=\pi/2$), equation $(5.7)_2$ reduces to equation $(4.6)_2$.

When the main reinforcement is not yielding, equations (5.5) and $(5.7)_1$ are solved to give:

(5.8) $\mu_1 = -\kappa\cot\gamma$ or $\Phi_1 = \nu/2$ and

(5.9) $\lambda\rho_1-\kappa = \sin\gamma-\lambda\cos\gamma$ or $\psi_1 = \frac{\nu}{2}\,\frac{\lambda(1+\cos\gamma)-\sin\gamma}{\sin^2\gamma(\lambda+\cot\gamma)}$

For $\rho\leq\rho_1$, the lowest upper bound is obtained with $\cot\beta=-\cot\gamma$. We determine α from equation $(5.2)_1$ and insert into equation (5.1) to get the solution:

$$\cot\beta = -\cot\gamma$$

(5.10) $$\tan\alpha = \frac{\mu+\kappa\cot\gamma}{\sqrt{1+\cot^2\gamma-(\mu+\kappa\cot\gamma)^2}} = \frac{\bar{\mu}}{\sqrt{1+\cot^2\gamma-\bar{\mu}^2}}$$

$$\tau/f_c = \frac{\nu\lambda}{2}\,\frac{\sqrt{1+\cot^2\gamma-(\mu+\kappa\cot\gamma)^2}+\cot\gamma}{\lambda+\cot\gamma} = \frac{\nu\lambda}{2}\,\frac{\sqrt{1+\cot^2\gamma-\bar{\mu}^2}+\cot\gamma}{\lambda+\cot\gamma}$$

Note that equation $(5.10)_2$ is identical with equation $(3.12)_2$.

Equations (5.10) are also obtained by inserting $\rho=\rho_1$ into equations (5.4), and using equations (3.3). For vertical stirrups ($\gamma=\pi/2$), the solution reduces to equations (2.10). From equation $(5.10)_2$ it appears that no yielding of the main reinforcement ($\alpha=o$) corresponds to $\mu=-\kappa\cot\gamma$, as found above. Inserting into equations (5.10), we get:

$$\cot\beta = -\cot\gamma$$

$$\text{(5.11)} \qquad \tan\alpha = o$$

$$\tau/f_c = \frac{\nu}{2}\,\frac{\lambda}{\lambda+\cot\gamma}\cot\frac{\gamma}{2}$$

Equations (5.11) are valid for $\mu\leq\mu_1$ and $\rho\leq\rho_1$, the limits being given by equations (5.8) and (5.9).

The condition $\cot\beta\leq\lambda$ is always satisfied, even for very small reinforcement degrees. Letting $\psi\rightarrow o$ and using equations (3.3), equation $(5.4)_1$ yields:

$$\cot\beta \rightarrow \frac{\lambda^2-(1-\mu^2)}{2\lambda}<\lambda \quad ,$$

as for vertical stirrups, cf. Section 4.1.

5.2 Discussion

Beams with distributed loading and inclined stirrups have not been treated before, consequently no lower bound solutions are known. It is straightforward, however, to construct a stress distribution similar to the one used in reference [3] in the case of vertical stirrups (cf. Section 4.2). Therefore there is no reason to doubt that when the longitudinal reinforcement is sufficiently strong to prevent yielding at failure ($\Phi \geq \Phi_1$), the equations $(5.6)_3$ and $(5.11)_3$ represent the complete solution, corresponding to yielding of the stirrups or not, respectively.

Vertical equilibrium along a concrete strut starting at the support (cf. Figure 5.2) yields:

$$V - bph\cot\theta = bhrf_y \sin^2\gamma(\cot\theta + \cot\gamma) \qquad \text{or}$$

$$(5.12) \qquad \lambda \frac{p}{f_c} - \frac{p}{f_c}\cot\theta = \psi \sin^2\gamma(\cot\theta + \cot\gamma)$$

The optimum strut inclination $\theta = \beta/2$ is found from equation $(5.6)_1$:

$$(5.13) \qquad \cot\theta = \frac{\sqrt{1+\lambda^2-(\lambda\rho-\kappa)^2}-1}{\lambda(1-\rho)+\kappa}$$

Comparing with equation (3.17) we note that the inclination of the strut (and hence also of the yield line) is steeper by distributed loading than by point loading for the same amount of shear reinforcement and stirrup inclination. On the other hand, comparing with equation (4.12), we see that to take account of the fact the stirrups are inclined, we replace the term $\lambda\rho$ by the quantity $\lambda\rho-\kappa$, where ρ is now given by equation $(3.3)_1$. This is equivalent with replacing $\lambda\psi$ by $\lambda\psi\sin^2\gamma + \psi\cos\gamma\sin\gamma$, which is smaller provided $\cot\gamma > 1/\lambda$. Since the strut inclination increases with increasing ψ, this means that the struts (and the yield lines) are the flatter the more inclined the stirrups are.

Parallel to the case of vertical stirrups (cf. Section 4.2) it turns out that the ultimate shear stress $\tau = \lambda p$ for distributed loading is obtained from the solution for point loading by replacing the parameter ψ by the quantity $\psi + p/f_c \sin^2\gamma$. Indeed, inserting into equation $(3.6)_3$:

$$\frac{\tau}{f_c} = \frac{\lambda p}{f_c} = \frac{\nu}{2(1-\kappa^2)}\left[\sqrt{(1-(\rho-\frac{2p}{\nu f_c})^2-\kappa^2)(1-\mu^2-\kappa^2)}+\kappa(1-\kappa^2+\mu(\rho-\frac{2p}{\nu f_c}))\right]$$

This equation may be solved to give equation $(5.4)_3$. Similarly, equation $(5.6)_3$ is obtained from equation $(3.8)_3$. Thus the reduction in stirrup steel when the loading is distributed rather than concentrated is even greater for inclined stirrups than for vertical. When the main reinforcement is not yielding ($\Phi \geq \Phi_1$), this fact is also evident from equation (5.12). Inserting equation (5.13), we again arrive at equation $(5.6)_3$. Thus, as for vertical stirrups, the shear reinforcement may be designed for a reduced load $V-bph\cot\theta$, but when the stirrups are inclined, the strut inclination θ may be chosen even flatter (cf. reference [14]).

The various upper bound solutions are summarized in Section 6, and the domains are sketched on Figure 6.4. For strong main reinforcement, the boundary between yielding and non-yielding of stirrups is $\psi=\psi_1$, given by equation (5.9). As for point loading, the value of ψ_1 increases with stirrup inclination. Comparing equations $(5.11)_3$ and $(4.8)_3$, we note that for strong shear reinforcement, inclined stirrups are more effective than vertical, as in the case of point loading, cf. equations $(3.13)_3$ and $(2.11)_3$. However, the enhancement factor is smaller for distributed loading, and it depends upon the shear span ratio λ.

When the main reinforcement is not strong enough to ensure a vertical displacement rate at failure, then the upper bound solution predicts a certain reduction in ultimate load. A corresponding lower bound is not easily constructed, but as in the other cases considered previously, the shear strength is not likely to be important in this range of parameter values.

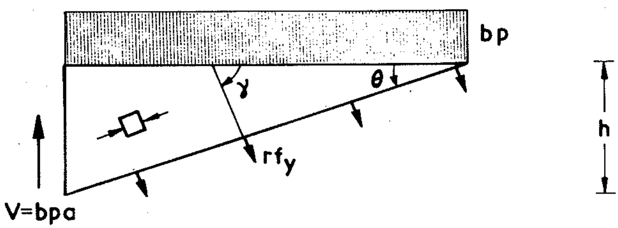


Fig. 5.2 : End zone of beam with inclined stirrups, bounded by section parallel to concrete compression

6. SUMMARY AND CONCLUSIONS

In the preceding sections, we have considered shear in simply supported beams with vertical or inclined stirrups, subjected to concentrated or distributed loading. Upper bound solutions have been derived for the shear strength corresponding to a failure mechanism with a yield line inclined at the angle β to the beam axis and a relative displacement rate inclined at the angle α to the beam normal.

The solutions are summarized below, on non-dimensional form. The shear span ratio is termed $\lambda=a/h$, and the degrees of shear and main reinforcement are defined as follows:

(2.4) $$\psi = \frac{rf_y}{f_c} \quad \text{and} \quad \Phi = \frac{T_y}{bhf_c}$$

We further introduce the auxiliary variables ρ, μ, $\bar{\mu}$, and κ, defined as:

(3.3) $$\rho = 1-\frac{2\psi}{\nu}\sin^2\gamma \quad , \quad \mu = 1-\frac{2\Phi}{\nu} - \frac{2\psi}{\nu}\cos^2\gamma$$

$$\bar{\mu} = 1-\frac{2\Phi}{\nu} \quad , \quad \kappa = \frac{2\psi}{\nu}\cos\gamma\sin\gamma$$

Here γ is the stirrup inclination.

<u>Concentrated loading, vertical stirrups</u>

Moderate main reinforcement : $\Phi \leq \nu/2$

Weak shear reinforcement : $\psi \leq \psi_o(\Phi)$, where

$(2.12)_2$ $$\psi_o = \frac{\nu}{2}\frac{\sqrt{\lambda^2+1-\mu^2}-\lambda}{\sqrt{\lambda^2+1-\mu^2}}$$

$$\cot\beta = \lambda$$

(2.13) $$\tan\alpha = \frac{\mu}{\sqrt{\lambda^2+1-\mu^2}}$$

$$\tau/f_c = \frac{\nu}{2}\left(\sqrt{\lambda^2+1-\mu^2}-\rho\lambda\right)$$

Moderate main reinforcement : $\Phi \leq \nu/2$

Moderate shear reinforcement : $\psi_o \leq \psi \leq \nu/2$

$$\cot\beta = \rho\sqrt{\frac{1-\mu^2}{1-\rho^2}}$$

(2.8) $$\tan\alpha = \mu\sqrt{\frac{1-\rho^2}{1-\mu^2}}$$

$$\tau/f_c = \frac{\nu}{2}\sqrt{(1-\mu^2)(1-\rho^2)}$$

Moderate main reinforcement : $\Phi \leq \nu/2$

Strong shear reinforcement : $\psi \geq \nu/2$

$$\cot\beta = o$$

(2.10) $$\tan\alpha = \frac{\mu}{\sqrt{1-\mu^2}}$$

$$\tau/f_c = \frac{\nu}{2}\sqrt{1-\mu^2}$$

Strong main reinforcement : $\Phi \geq \nu/2$

Weak shear reinforcement : $\psi \leq \frac{\nu}{2}\,\frac{\sqrt{1+\lambda^2}-\lambda}{\sqrt{1+\lambda^2}}$

$$\cot\beta = \lambda$$

(2.14) $$\tan\alpha = o$$

$$\tau/f_c = \frac{\nu}{2}\left(\sqrt{\lambda^2+1}-\rho\lambda\right)$$

Strong main reinforcement $\Phi \geq \nu/2$

Moderate shear reinforcement : $\frac{\nu}{2}\,\frac{\sqrt{1+\lambda^2}-\lambda}{\sqrt{1+\lambda^2}} \leq \psi \leq \nu/2$

$$\cot\beta = \frac{\rho}{\sqrt{1-\rho^2}}$$

(2.9) $$\tan\alpha = o$$

$$\tau/f_c = \frac{\nu}{2}\sqrt{1-\rho^2}$$

Strong main reinforcement : $\Phi \geq \nu/2$

Strong shear reinforcement : $\psi \geq \nu/2$

$$\cot\beta = o$$

(2.11) $$\tan\alpha = o$$

$$\tau/f_c = \nu/2$$

Figure 6.1 shows the domains of the different solutions in the (ψ,Φ) parameter plane. The boundaries are plotted for a shear span ratio of $\lambda=2$. The corresponding failure mechanism is sketched for each domain.

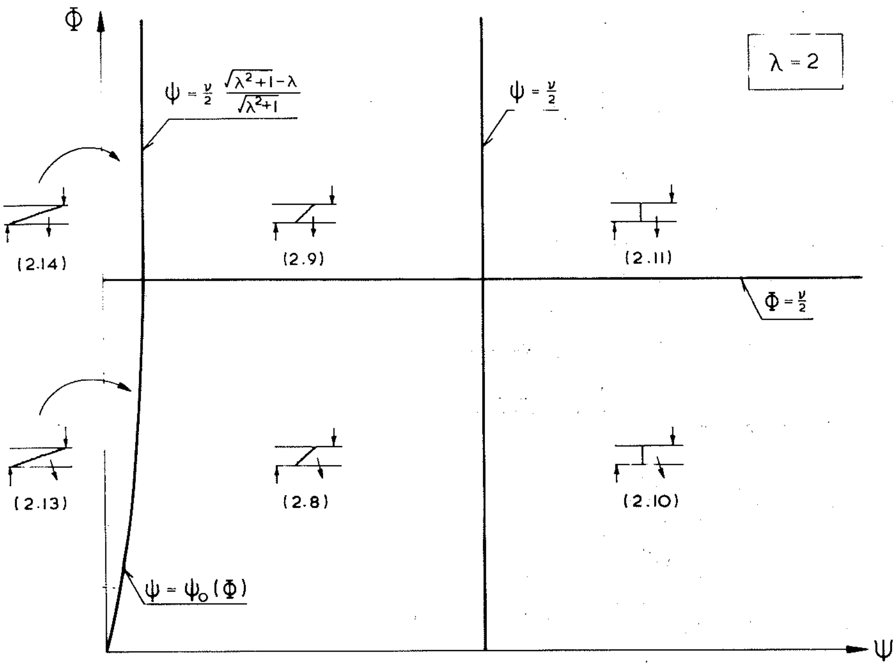


Fig. 6.1 : Domains of upper bound solutions for beams with vertical stirrups, subjected to point loading
$\psi=\psi_o(\Phi)$ given by equation (2.12)

Concentrated loading, inclined stirrups

Moderate main reinforcement : $\Phi \le \frac{\nu}{2} - \psi(\lambda+\cot\gamma)\cos\gamma\sin\gamma$

Weak Shear reinforcement : $\psi \le \psi_o(\Phi)$, where

$$(3.14)_2 \qquad \mu = \frac{-\lambda\kappa(1-\kappa^2-\rho^2)+\rho\sqrt{(1+\lambda^2)(\kappa^2+\rho^2)-\lambda^2}}{\kappa^2+\rho^2}$$

determines $\psi=\psi_o$.

$$(3.15) \qquad \begin{aligned} \cot\beta &= \lambda \\ \tan\alpha &= \frac{\mu-\lambda\kappa}{\sqrt{1+\lambda^2-(\mu-\lambda\kappa)^2}} \\ \tau/f_c &= \frac{\nu}{2}\left(\sqrt{1+\lambda^2-(\mu-\lambda\kappa)^2}-\rho\lambda+\kappa\right) \end{aligned}$$

Moderate main reinforcement : $\Phi \le \Phi_1(\psi)$

Moderate shear reinforcement : $\psi_o(\Phi) \le \psi \le \psi_1(\Phi)$, where

$$(3.7)_2 \qquad \Phi_1 = \frac{\nu}{2} - \psi\cos\gamma\left(\cos\gamma+\frac{\nu-2\psi\sin^2\gamma}{2\sqrt{\psi(\nu-\psi\sin^2\gamma)}}\right)$$

$$(3.9)_2 \qquad \psi_1 = \frac{\nu}{2}\left(1+\cot\gamma\,\frac{\cot\gamma+\sqrt{1+\cot^2\gamma-\bar{\mu}^2}}{1+\bar{\mu}}\right)$$

$$(3.6) \qquad \begin{aligned} \cot\beta &= \frac{1}{1-\kappa^2}\left[\rho\sqrt{\frac{1-\mu^2-\kappa^2}{1-\rho^2-\kappa^2}}-\kappa\mu\right] \\ \tan\alpha &= \frac{1}{1-\kappa^2}\left[\mu\sqrt{\frac{1-\rho^2-\kappa^2}{1-\mu^2-\kappa^2}}-\kappa\rho\right] \\ \tau/f_c &= \frac{\nu}{2(1-\kappa^2)}\left[\sqrt{(1-\rho^2-\kappa^2)(1-\mu^2-\kappa^2)}+\kappa(1-\kappa^2+\mu\rho)\right] \end{aligned}$$

Moderate main reinforcement : $\Phi \le \nu/2$

Strong shear reinforcement : $\psi \ge \psi_1(\Phi)$

$$(3.12) \qquad \begin{aligned} \cot\beta &= -\cot\gamma \\ \tan\alpha &= \frac{\bar{\mu}}{\sqrt{1+\cot^2\gamma-\bar{\mu}^2}} \\ \tau/f_c &= \frac{\nu}{2}\left(\sqrt{1+\cot^2\gamma-\bar{\mu}^2}+\cot\gamma\right) \end{aligned}$$

Strong main reinforcement : $\Phi \geq \frac{\nu}{2} - \psi(\lambda+\cot\gamma)\cos\gamma\sin\gamma$

Weak shear reinforcement : $\psi \leq \frac{\nu}{2} \frac{\sqrt{\lambda^2+1}-\lambda}{\sin^2\gamma\sqrt{\lambda^2+1}}$

$$\cos\beta = \lambda$$

(3.18) $$\tan\alpha = o$$

$$\tau/f_c = \frac{\nu}{2}\left(\sqrt{1+\lambda^2}-\rho\lambda+\kappa\right)$$

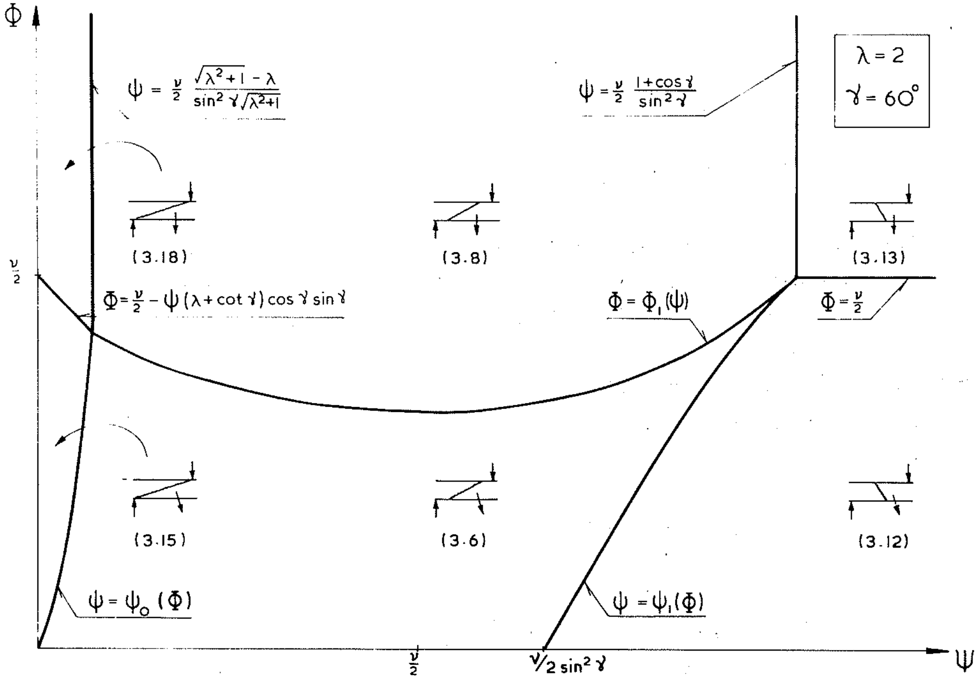


Fig. 6.2 : Domains of upper bound solutions for beams with inclined stirrups, subjected to point loading

$\psi=\psi_o(\Phi)$ given by equation (3.14)

$\psi=\psi_1(\Phi)$ given by equation (3.9)

$\Phi=\Phi_1(\psi)$ given by equation (3.7)

Strong main reinforcement : $\Phi \geq \Phi_1(\psi)$

Moderate shear reinforcement : $\frac{\nu}{2} \frac{\sqrt{1+\lambda^2}-\lambda}{\sin^2\gamma \sqrt{\lambda^2+1}} \leq \psi \leq \frac{\nu}{2} \frac{1+\cos\gamma}{\sin^2\gamma}$

(3.8)
$$\cot\beta = \frac{\rho}{\sqrt{1-\rho^2}}$$
$$\tan\alpha = 0$$
$$\tau/f_c = \frac{\nu}{2}\left(\sqrt{1-\rho^2}+\kappa\right)$$

Strong main reinforcement : $\Phi \geq \nu/2$

Strong shear reinforcement : $\psi \geq \frac{\nu}{2} \frac{1+\cos\gamma}{\sin^2\gamma}$

(3.13)
$$\cot\beta = -\cot\gamma$$
$$\tan\alpha = 0$$
$$\tau/f_c = \frac{\nu}{2} \cot\frac{\gamma}{2}$$

The domains of the solutions are shown on Figure 6.2. The boundaries are plotted in the case of a shear span ratio of $\lambda=2$ and a stirrup inclination of $\gamma=60^\circ$. For each domain, the corresponding failure mechanism is sketched.

Distributed loading, vertical stirrups

Moderate main reinforcement : $\Phi \leq \nu/2$

Weak and moderate shear reinforcement : $\psi \leq \psi_1(\Phi)$, where

(4.6)$_2$
$$\psi_1 = \frac{\nu}{2} \frac{\lambda-\sqrt{1-\mu^2}}{\lambda}$$

(4.4)
$$\cot\beta = \frac{1-\mu^2}{\lambda(1-\rho^2)}\left[\rho\sqrt{1+\lambda^2 \frac{1-\rho^2}{1-\mu^2}} - 1\right]$$
$$\tan\alpha = \frac{\lambda\mu}{1+\lambda^2-\mu^2}\left[\rho+\sqrt{1+\lambda^2 \frac{1-\rho^2}{1-\mu^2}}\right]$$
$$\tau/f_c = \frac{\nu\lambda(1-\mu^2)}{2(1+\lambda^2-\mu^2)}\left[\rho + \sqrt{1+\lambda^2 \frac{1-\rho^2}{1-\mu^2}}\right]$$

Moderate main reinforcement: $\Phi \leq \nu/2$

Strong shear reinforcement : $\psi \geq \psi_1(\Phi)$

$$\cot\beta = o$$

(4.7) $$\tan\alpha = \frac{\mu}{\sqrt{1-\mu^2}}$$

$$\tau/f_c = \frac{\nu}{2}\sqrt{1-\mu^2}$$

Strong main reinforcement : $\Phi \geq \nu/2$

Weak and moderate shear reinforcement : $\psi \leq \nu/2$

$$\cot\beta = \frac{1}{\lambda(1-\rho^2)}\left(\rho\sqrt{1+\lambda^2(1-\rho^2)}-1\right)$$

(4.5) $$\tan\alpha = o$$

$$\tau/f_c = \frac{\nu\lambda}{2(1+\lambda^2)}\left(\rho+\sqrt{1+\lambda^2(1-\rho^2)}\right)$$

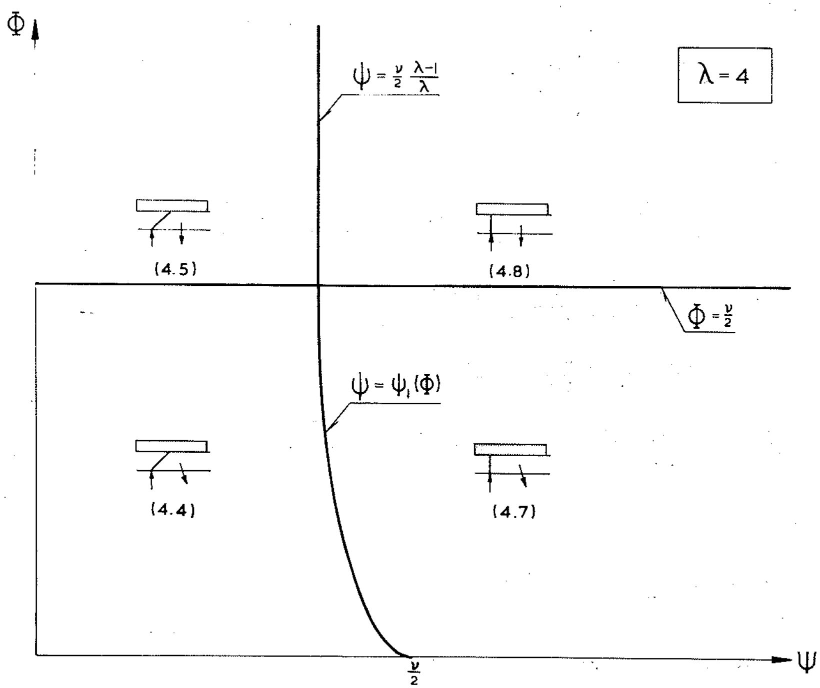


Fig. 6.3 : Domains of upper bound solutions for beams with vertical stirrups, subjected to distributed loading $\psi=\psi_1(\Phi)$ given by equation (4.6)

Strong main reinforcement : $\Phi \geq \nu/2$

Strong shear reinforcement : $\psi \geq \nu/2$

(4.8)
$$\cot\beta = 0$$
$$\tan\alpha = 0$$
$$\tau/f_c = \nu/2$$

Figure 6.3 shows the domains of the solutions, the boundaries being plotted for a shear span ratio of $\lambda=4$. The corresponding failure mechanisms are sketched for each domain.

Distributed loading, inclined stirrups

Moderate main reinforcement : $\Phi \leq \Phi_1(\psi)$

Weak and moderate shear reinforcement : $\psi \leq \psi_1(\psi)$, where

(5.5)
$$\mu_1 = \kappa \frac{(\lambda\rho-\kappa)\sqrt{1+\lambda^2-(\lambda\rho-\kappa)^2}-\lambda}{\lambda^2-(\lambda\rho-\kappa)^2}$$

determines Φ_1, and

$(5.7)_2$
$$\psi_1 = \frac{\nu}{2}\frac{(\lambda+\bar{\mu}\cot\gamma)(1+\bar{\mu})+\lambda\cot^2\gamma-(1+\bar{\mu}-\lambda\cot\gamma)\sqrt{1+\cot^2\gamma-\bar{\mu}^2}}{(1+\bar{\mu})(\lambda+\cot\gamma)}$$

(5.4)
$$\cot\beta = \frac{(\lambda\rho-\kappa)\sqrt{1+\lambda^2\eta}-\lambda-\kappa\mu(1+\lambda^2\eta)}{\lambda^2\eta-\kappa^2(1+\lambda^2\eta)}$$
$$\tan\alpha = \frac{(\lambda\mu+\kappa)\sqrt{1+\lambda^2\eta}+(\mu-\lambda\kappa)(\lambda\rho-\kappa)}{1+\lambda^2-(\mu-\lambda\kappa)^2}$$
$$\tau/f_c = \frac{\nu\lambda}{2}\frac{\lambda\kappa+\rho-(\mu-\lambda\kappa)(\mu\rho-\kappa^2)+(1-\mu^2-\kappa^2)\sqrt{1+\lambda^2\eta}}{1+\lambda^2-(\mu-\lambda\kappa)^2}$$

where

(5.3)
$$\eta = \frac{\lambda(1-\rho^2-\kappa^2)+2\kappa(\mu+\rho)}{\lambda(1-\mu^2-\kappa^2)}$$

Moderate main reinforcement : $\Phi \leq \nu/2$

Strong shear reinforcement : $\psi \geq \psi_1(\Phi)$

$$\cot\beta = -\cot\gamma$$

(5.10) $$\tan\alpha = \frac{\bar{\mu}}{\sqrt{1+\cot^2\gamma-\bar{\mu}^2}}$$

$$\tau/f_c = \frac{\nu\lambda}{2}\,\frac{\sqrt{1+\cot^2\gamma-\bar{\mu}^2}+\cot\gamma}{\lambda+\cot\gamma}$$

Strong main reinforcement : $\Phi \geq \Phi_1(\psi)$

Weak and moderate shear reinforcement : $\psi \leq \frac{\nu}{2}\,\frac{\lambda(1+\cos\gamma)-\sin\gamma}{\sin^2\gamma(\lambda+\cot\gamma)}$

$$\cot\beta = \frac{(\lambda\rho-\kappa)\sqrt{1+\lambda^2-(\lambda\rho-\kappa)^2}-\lambda}{\lambda^2-(\lambda\rho-\kappa)^2}$$

(5.6) $$\tan\alpha = 0$$

$$\tau/f_c = \frac{\nu\lambda}{2}\,\frac{\rho+\lambda\kappa+\sqrt{1+\lambda^2-(\lambda\rho-\kappa)^2}}{1+\lambda^2}$$

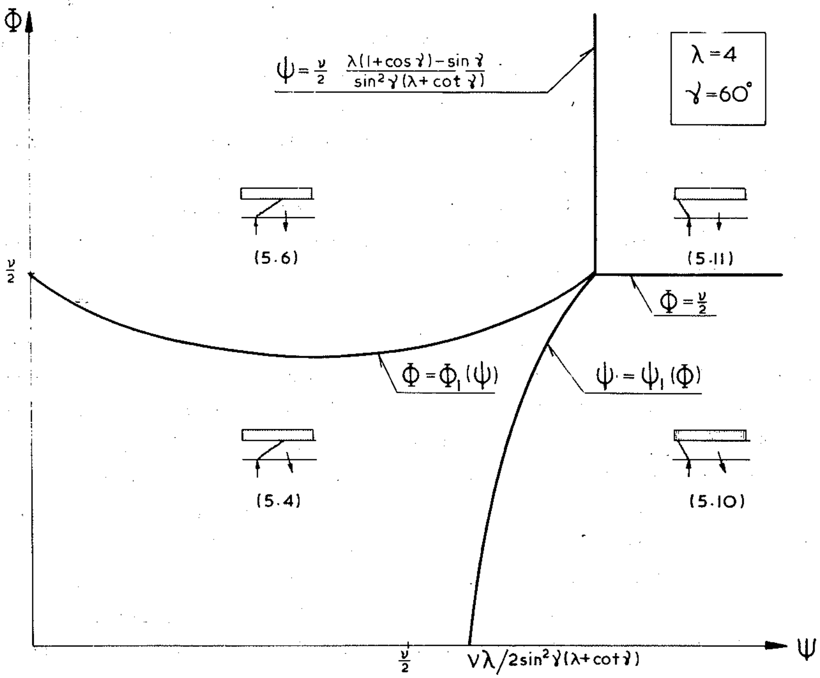


Fig. 6.4 : Domains of upper bound solutions for beams with inclined stirrups, subjected to distributed loading

$\psi=\psi_1(\Phi)$ given by equation (5.7)

$\Phi=\Phi_1(\psi)$ given by equation (5.5)

Strong main reinforcement : $\Phi \geq \nu/2$

Strong shear reinforcement : $\psi \geq \frac{\nu}{2}\,\frac{\lambda(1+\cos\gamma)-\sin\gamma}{\sin^2\gamma(\lambda+\cot\gamma)}$

$$\cot\beta = -\cot\gamma$$

$$\tan\alpha = o \qquad (5.11)$$

$$\tau/f_c = \frac{\nu}{2}\,\frac{\lambda}{\lambda+\cot\gamma}\cot\frac{\gamma}{2}$$

Figure 6.4 shows the domains of the solutions. The boundaries are plotted for a shear span ratio of $\lambda=4$ and a stirrup inclination of $\gamma=60^0$. For each domain, the corresponding failure mechanism is sketched.

The solutions corresponding to strong main reinforcement are complete, i.e. identical with a lower bound. The strut inclination θ of a corresponding stress distribution (diagonal compression field) is determined as $\theta=\beta/2$. The struts are steeper by distributed than by point loading, and (generally) flatter by inclined than by vertical stirrups. For strong shear reinforcement, inclined stirrups are more effective than vertical, in the sense that the same amount of web reinforcement leads to higher shear strength. The ultimate shear load is greater by distributed loading than by point loading. Indeed, the same maximum shear stress is maintained when the shear reinforcement degree is reduced by the amount $p/f_c\sin^2\gamma$, p being the intensity (per unit area) of the distributed load. Equivalently, the stirrup reinforcement may be designed for the shear force V−bphcotθ, i.e. the shear force in the distance hcotθ from the support.

When the main reinforcement is not sufficiently strong, it will yield at shear failure of the beam. The upper bound solutions predict a reduction of the shear strength, but no corresponding lower bounds are indicated, except for beams without any stirrup reinforcement. The reduction is likely to be of limited importance, because a substantial main reinforcement is required in order to prevent flexural failure. A possible exception are beams with little shear reinforcement, and for such cases, lower bound solutions are developed (JENSEN & al.[11]). Solutions corresponding to weak main reinforcement are also of interest in connection with the shear strength of beams subjected to additional axial loading. Finally they may explain premature collapses due to anchorage failure, which reduces the main reinforcement strength available in shear without significantly affecting the flexural strength.

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