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Averaging Methods in Random Vibration

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AVERAGING METHODS IN RANDOM VIBRATION

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Abstract

This text represents a series of lectures given by the author in August-September 1988 during a visiting professorship at the Technical University of Denmark, Department of Structural Engineering. The subject is stochastic averaging methods in random vibration, a field initiated in the early sixties and through subsequent refinements it has provided very powerfull tools in random vibration analysis. The main requirement of the method is that the "memory" of the system considered is much larger than the "memory" present in the random excitation. In terms of a linear oscillator this corresponds to a narrow band transfer function (i.e. small damping) compared to the excitation spectrum. The subject is treated from the viewpoint of an engineer relying on physical rather than mathematical arguments so even readers unfamiliar with random vibration analysis may find the text usefull.

Chapter one contains a brief introduction to random vibration analysis giving examples of physical systems where nonlinearities play a significant role. Some frequently used solution methods are briefly discussed. In chapter two attention is directed towards the basic averaging method for single degree of freedom systems with nonlinear damping and/or parametric excitation. In chapter three the averaging method is extended to multi-degree of freedom systems, non-stationary excitation and response and systems with non-linearities in the stiffness term. The first-passage problem is adressed in chapter four, discussing first-passage problems in general and the different solution techniques and closing with a more extensive treatment using the averaging approximation. In chapter five two practical examples are studied. The first is roll motion of a ship in random waves where damping and stiffness non-linearities and parametric effects are present. Secondly hysteretic oscillators are treated and for both examples the averaging method is shown to be very efficient.

References are listed at the end of each chapter and in addition to some of the most accepted basic references in random vibration, a large number of references considering stochastic averaging are provided. For a more detailed treatment of the topics covered in this text, the author often refers to publications of his own.

Resumé

Denne rapport er skrevet som supplerende materiale til en forelæsningsserie præsenteret af forfatteren i august-september 1989 under et ophold som gæsteprofessor ved Danmarks Tekniske Højskole, Afdelingen for Bærende Konstruktioner. Emnet er metoder til "stokastisk midling" indenfor stokastisk dynamik. Fremgangsmåden blev foreslået først i 60'erne og er siden generaliseret og har resulteret i meget effektive analyseværktøjer indenfor stokastisk dynamik. Den væsentligste forudsætning for metodens anvendelighed er at "hukommelsen" i det betragtede dynamiske system er længere end "hukommelsen" i den stokastiske belastning. For en simpel oscillator er dette ensbetydende med en smalbåndet frekvensresponsfunktion (dvs. lille dæmpning) set i relation til belastningsspektret. Forfatteren anvender en ingeniørs synsvinkel, hvor de fysiske argumenter vægtes højere end de matematiske, så selvom læseren ikke har indgående kendskab til stokastisk dynamik vil nærværende tekst vise sig interessant, forståelig og nyttig.

I kapitel 1 gives en kort introduktion til stokastisk dynamik med eksempler på aktuelle fysiske systemer, hvor ulineariteter er af betydning. I denne sammenhæng gives en kort oversigt over de mest anvendte analysemetoder. I kapitel 2 gives en grundig redegørelse for "stokastisk midling" i sin simple oprindelige form, hvor der betragtes et 1-frihedsgradssystem med ulineær dæmpning og/eller parametrisk belastning. I kapitel 3 udvides metoden til fler-frihedsgrads systemer, ikkestationært respons og/eller belastning samt systemer med ulineær stivhed. Første-passage problemet behandles i kapitel 4, hvor der indledningsvis gives en generel introduktion til emnet og de sædvanlige analysemetoder, hvorefter anvendelsen af stokastisk midling gennemgås mere detailleret. Afslutningsvis gives der i kapitel 5 to eksempler på praktiske problemer, hvor "stokastisk midling" er bragt i anvendelse. Det første drejer sig om krængningsbevægelser af skibe i søgang, hvor dæmpning og stivhed er ulineære og der endvidere kan forekomme parametrisk belastning. Det sidste eksempel er oscillatorer med hystereseeffekt i tilbageføringskraften enten i form af bilineær hysterese eller en mere realistisk differentiabel hysteresekarakteristik. I begge tilfælde viser "stokastisk midling" sig meget effektiv.

Referencerne er opgivet i slutningen af hvert kapitel og foruden nogle væsentlige grundlæggende værker om stokastisk dynamik, er der her listet en lang række artikler specielt om stokastisk midling. Søges en grundigere redegørelse for resultaterne angivet i denne rapport, henviser forfatteren ofte til egne publikationer.

CONTENTS

AVERAGING METHODS IN RANDOM VIBRATION

J. B. Roberts

CHAPTER ONE

Random Vibration Problems: Sources of Non-linearity, Methods of solution

1. General statement of the problem

In this text we will be concerned with dynamic systems driven by an excitation process, $X(t)$; this results in a response process, $Y(t)$, as shown in Fig. 1.1. It will be assumed that sample functions of $X(t)$ are complex functions of time, such that $X(t)$ is best modelled as a stochastic (or random process). It follows that $Y(t)$ is also a stochastic process, with complex sample functions.

The general random vibration problem may be stated as follows:

To predict analytically, the probabilistic behaviour of the response from a knowledge of

(a) The statistics of the excitation, and
(b) the equation of motion.

1.1 Linear systems

If the system is linear, and the excitation is Gaussian, then the response is also Gaussian. It is then sufficient to relate the mean and covariance of the response to the mean and covariance of the input (e.g. see Lin (1967)). Standard theory

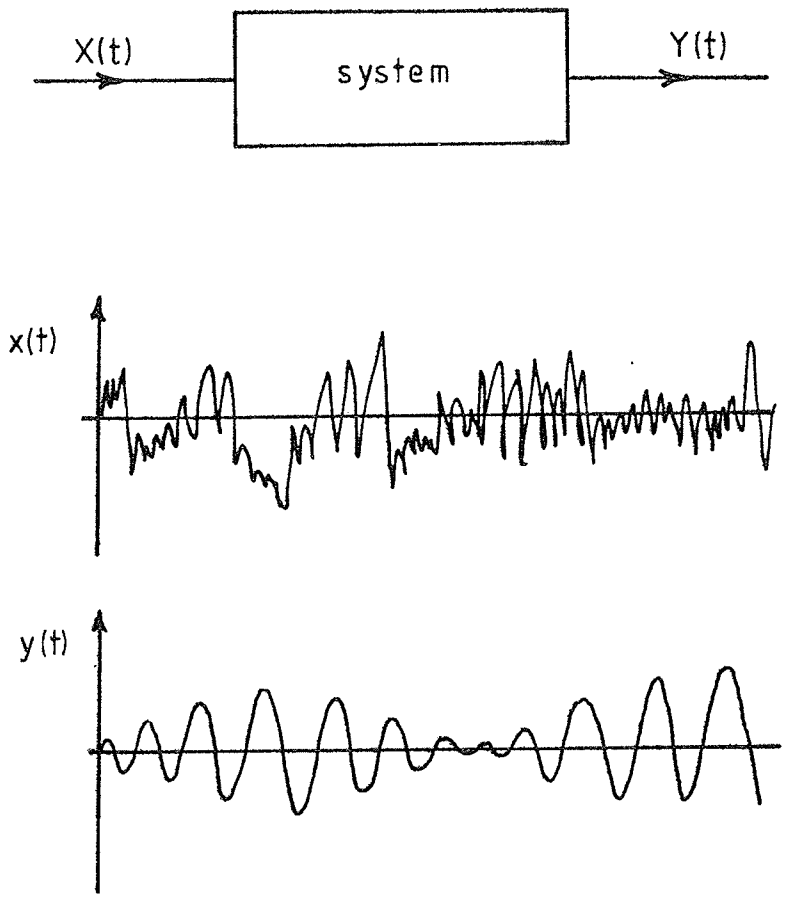


Fig. 1.1.

exists for this purpose and may be found in numerous text-books (e.g. Crandall and Mark (1964), Robson (1963), Lin (1967), Newland (1978), Nigam (1983), Yang (1986)).

The problem is more difficult if the excitation is non-Gaussian. It is possible to develop a set of moment equations relating the excitation to the response, from which the distribution of the response may be studied, using expressions such as the Gram-Charlier series (e.g. see Stratonovich (1964)). However, it is worth remarking that lightly damped systems have a tendency to transform a non-Gaussian process into a more closely Gaussian process, as can be seen through an application of the Central Limit Theorem. Thus, in most cases, response processes can be treated as Gaussian, to a good approximation if the system is linear.

1.2 Non-linear systems

When non-linearities are present in the system, the probability distribution of the response can be highly non-Gaussian. For example, a "hardening" spring can inhibit the growth of large amplitude response, resulting in large departures from Gaussianity in the "tails" of the distribution. The effect is illustrated in Fig. 1.2, which sketches a typical comparison between a Gaussian probability density function (plotted on a log-linear basis) and a corresponding typical non-Gaussian function, relating to a response process.

In many practical applications one is concerned with estimating reliability statistics. For example one may wish to estimate the probability that the response stays within some safe region, during some specified interval of time (this is the so-called "first-passage problem", which will be discussed in some detail, in a later lecture). Such reliability statistics are intimately connected with the behaviour of the probability distribution of the response, at large amplitudes. Clearly, then, to obtain meaningful reliability estimates it is vital to account for the deviations from Gaussianity, due to non-linearity.

1.3 Applications

The problem of predicting the response of non-linear systems to random excitation, with a view to designing such systems to operate reliabily, and safely, occurs in many fields of engineering. Here we list a few of the main areas of application.

(a) response of aero-space vehicles to atmospheric turbulence, jet noise, etc.

(b) response of civil engineering structures, such as buildings, bridges, dams, etc., to earthquake excitation, and to wind loading.

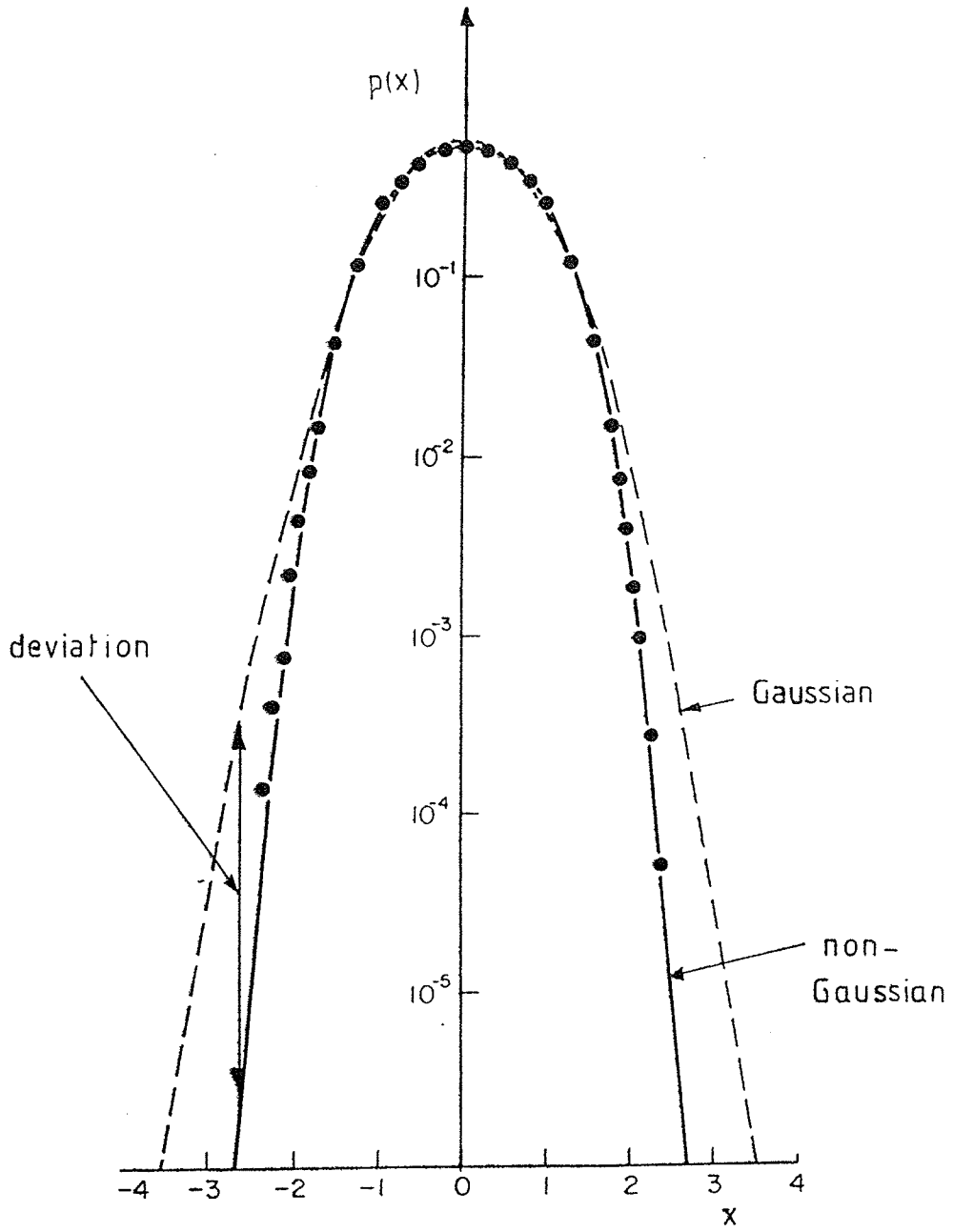


Fig. 1.2.

(c) response of marine vehicles and offshore platforms to wave excitation and to wind loading.

(d) response of land vehicles (e.g. trains, cars, etc.) to irregularities in the ground surface on which they travel.

2. The general equations of motion

To begin with, we consider the formulation of the equations of motion of a system which has been discretised (in a suitable fashion) into an n-degree of freedom, lumped-parameter system. It will be assumed throughout that the system is stable and time-invariant.

Let the motion be described by n generalised coordinates, $q_1, q_2, \ldots, q_n$. The displacement vector, $\underset{\sim}{q}$, may be defined as

$$\underset{\sim}{q} = [q_1, q_2, \ldots, q_n]^T \tag{1}$$

A small, virtual displacement of the system can be described by $\delta\underset{\sim}{q}$. The forces acting on the system (including inertial forces) will do a certain amount of work, δW, during this displacement. Thus,

$$\delta W = \underset{\sim}{Q}^T \delta\underset{\sim}{q} \tag{2}$$

where $\underset{\sim}{Q}$ is the n-vector of generalised forces corresponding to $\underset{\sim}{q}$. By the Principle of Virtual Work

$$\delta W = 0 \tag{3}$$

It follows that

$$Q_i = 0 \qquad (i = 1,2,\ldots,n) \tag{4}$$

where Q_i are the elements of $\underset{\sim}{Q}$. Equation (4) represents the equations of motion of the system.

It is convenient to decompose $\underset{\sim}{Q}$ as follows:

$$\underset{\sim}{Q} = \underset{\sim}{Q}^I + \underset{\sim}{Q}^R + \underset{\sim}{Q}^D + \underset{\sim}{Q}^E \tag{5}$$

where

$\underset{\sim}{Q}^I$ - inertial force vector

$\underset{\sim}{Q}^R$ - restoring force vector

$\underset{\sim}{Q}^D$ - dissipative (or damping) force vector

$\underset{\sim}{Q}^E$ - external (applied) force vector

The equations of motion can now be written as

$$Q_i^I + Q_i^R + Q_i^D + Q_i^E = 0 \qquad (i = 1,2,\ldots,n) \tag{6}$$

2.1 Inertial forces

Q_i^I can be found from the kinetic energy function T, using the usual Lagrangian expression (e.g. see Whittaker (1937)). Thus

$$Q_i^I = -\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_i}\right] + \frac{\partial T}{\partial q_i} \tag{7}$$

In general T is a quadratic function of the generalized velocities; thus

$$T = \tfrac{1}{2}(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + a_{nn}\dot{q}_n^2 + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots) \tag{8}$$

where the coefficients a_{11}, etc., in general depend on $\underset{\sim}{q}$. In matrix form

$$T = \frac{1}{2}\,\dot{\underset{\sim}{q}}^T\,\underset{\approx}{A}\,\dot{\underset{\sim}{q}} \tag{9}$$

where $\underset{\approx}{A}$ is the $n\times n$ "inertia matrix".

Substituting into equation (7) from equation (8) gives

$$Q_i^I = -\sum_{j=1}^{n} a_{ij}\ddot{q}_j - \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial a_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j$$

$$+ \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial a_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k \qquad (10)$$

2.2 Restoring forces

For *conservative* (i.e., path independent) forces one can introduce the potential energy function

$$V = V(q_1, q_2, \ldots, q_n) \qquad (11)$$

and it is easy to show that

$$Q_i^R = -\frac{\partial V}{\partial q_i} \qquad (12)$$

2.3 Dissipative forces

The dissipative forces, Q_i^D , can be decomposed into the sum of two components; thus

$$Q_i^D = Q^{D\ell}{}_i + Q_i^* \qquad (13)$$

where

$Q^{D\ell}{}_i$ - linear viscous component

Q_i^* - other types of damping.

For linear viscous damping it is useful to introduce the Rayleigh dissipation function

$D = \frac{1}{2}$ total rate of energy loss

$$= \frac{1}{2} (b_{11}\dot{q}_1^2 + b_{22}\dot{q}_2^2 + \ldots + 2b_{12}\dot{q}_1\dot{q}_2 + \ldots) \tag{14}$$

or, in matrix form,

$$D = \frac{1}{2} \underset{\sim}{\dot{q}}^T \underset{\sim}{B} \underset{\sim}{\dot{q}} \tag{15}$$

where $\underset{\sim}{B}$ is an $n \times n$ "damping matrix" of constants. $Q_i^{D\ell}$ may be derived from D as follows:

$$Q_i^{D\ell} = - \frac{\partial D}{\partial \dot{q}_i} \tag{16}$$

Hence

$$Q_i^{D\ell} = - \sum_{j=1}^{n} b_{ij} \dot{q}_j \tag{17}$$

More generally

$$Q_i^{D} = - \sum_{j=1}^{n} b_{ij} \dot{q}_j + Q_i^* \tag{18}$$

2.4 Combined results

Collecting the above results one has

$$\sum_{j=1}^{n} a_{ij} \ddot{q}_j + \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial a_{ij}}{\partial q_k} \dot{q}_j \dot{q}_k$$

$$-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial a_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k + \sum_{j=1}^{n} b_{ij}\dot{q}_j - Q_i^* + \frac{\partial V}{\partial q_j} = Q_i^E \qquad (19)$$

This is a set of n coupled, non-linear differential equations. Sources of non-linearity are

(a) dependency of inertial coefficients, a_{ij} , on $\underset{\sim}{q}$

(b) form of non-viscous damping forces, Q_i^*

(c) form of the potential energy function, V .

For the special case of a single degree of freedom system (n=1) , equation (19) reduces to (omitting subscripts)

$$a(q)\ddot{q} + \frac{1}{2}\frac{\partial a}{\partial q}\dot{q}^2 + b\dot{q} - Q^* + \frac{\partial V}{\partial q} = Q^E \qquad (20)$$

2.5 Linear approximation for small amplitude motion

For small amplitude motion about a position of static equilibrium ($\underset{\sim}{q} = \underset{\sim}{0}$) , useful approximations can be made.

The elements of the inertia matrix may be treated as constants, if the amplitude of motion is sufficiently small. Thus, expanding $a_{11}(\underset{\sim}{q})$ about $\underset{\sim}{q} = \underset{\sim}{0}$ one has

$$a_{11}(\underset{\sim}{q}) = a_{11}(\underset{\sim}{0}) + \text{higher order terms} \qquad (21)$$

A linear approximation is to replace $a_{11}(\underset{\sim}{q})$ by $a_{11}(\underset{\sim}{0})$, and similarly for all the other elements of $\underset{\approx}{A}$.

The potential energy function $\underset{\sim}{V}$ can also be approximated. Thus

$$V(\underset{\sim}{q}) = \frac{1}{2}(k_{11}q_1^2 + k_{22}q_2^2 + \ldots + 2k_{12}q_1q_2 + \ldots)$$

$$+ \text{higher order terms in } q_1, q_2, \text{etc.} \qquad (22)$$

If the higher order terms in $V(\underset{\sim}{q})$ are neglected one has

$$\underset{\sim}{V} = \frac{1}{2} \underset{\sim}{q}^T \underset{\sim}{K} \underset{\sim}{q} \tag{23}$$

where $\underset{\sim}{K}$ is an $n \times n$ "stiffness matrix", the elements of which are constants.

Hence

$$Q_i^R = - \sum_{j=1}^{n} k_{ij} q_j \tag{24}$$

and, collecting results, and ignoring Q_i^*, one has the well-known matrix equation of motion

$$\underset{\sim}{A} \ddot{\underset{\sim}{q}} + \underset{\sim}{B} \dot{\underset{\sim}{q}} + \underset{\sim}{K} \underset{\sim}{q} = \underset{\sim}{Q} \tag{25}$$

where the superscript E has now been dropped.

2.6 Equations for non-linear motion

If the non-linear contributions to the inertia, stiffness and damping components are not neglected (i.e. large amplitude motion prevails) then instead of equation (25) one can write

$$\underset{\sim}{A} \ddot{\underset{\sim}{q}} + \underset{\sim}{B} \dot{\underset{\sim}{q}} + \underset{\sim}{K} \underset{\sim}{q} + \underset{\sim}{\Phi}(\underset{\sim}{q}, \dot{\underset{\sim}{q}}, \ddot{\underset{\sim}{q}}) = \underset{\sim}{Q} \tag{26}$$

where $\underset{\sim}{\Phi}(\underset{\sim}{q}, \dot{\underset{\sim}{q}}, \ddot{\underset{\sim}{q}})$ is derived from all the sources of non-linearity.

In some problems $\underset{\sim}{\Phi}$ depends on the history of the motion, and one can not just treat it as an instantaneous function of $\underset{\sim}{q}$, $\dot{\underset{\sim}{q}}$ and $\ddot{\underset{\sim}{q}}$ (as indicated above). This occurs when the restoring force is hysteretic in nature. Methods of dealing with hysteretic systems will be discussed later.

3. Examples of non-linear conservative restoring forces

We now discuss briefly a number of specific examples, where non-linearity occurs in conservative restoring forces.

3.1 Motion in a gravitational field

A very simple example is furnished by the pendulum, executing planar motion. If θ is the angular displacement from the equilibrium position (see Fig. 1.3(a)) then

$$V = mgl(1 - \cos\theta) \tag{27}$$

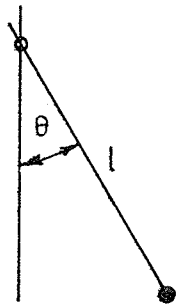
and $$\frac{\partial V}{\partial \theta} = mgl \sin\theta \tag{28}$$

Thus applying equation (20), one has (in the case of no damping)

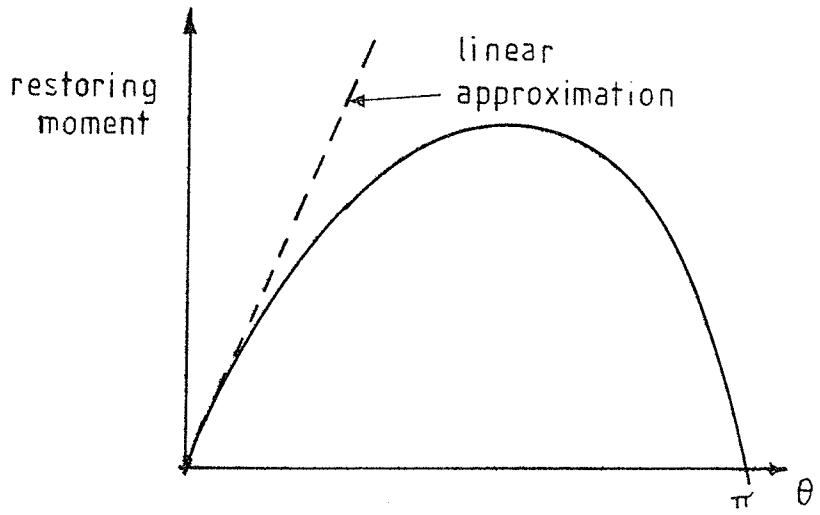
$$(ml^2)\ddot{\theta} + mgl \sin\theta = 0 \tag{29}$$

for free vibration. A linearised approximation is to replace $\sin\theta$ by θ (see Fig. 1.3(b)). However, this approximation is unjustifiable for large amplitude motion.

It is noted that, in the phase-plane (see Fig. 1.3(c)), there is only a finite region where oscillatory motion is possible. This poses some difficulty if the system is excited by stationary random excitation, since the response will eventually exit from the oscillatory region. Thus care must be taken when dealing with systems where the non-linearity in stiffness is of the softening type. This matter will be returned to later, in the context of ship roll motion.



(a)



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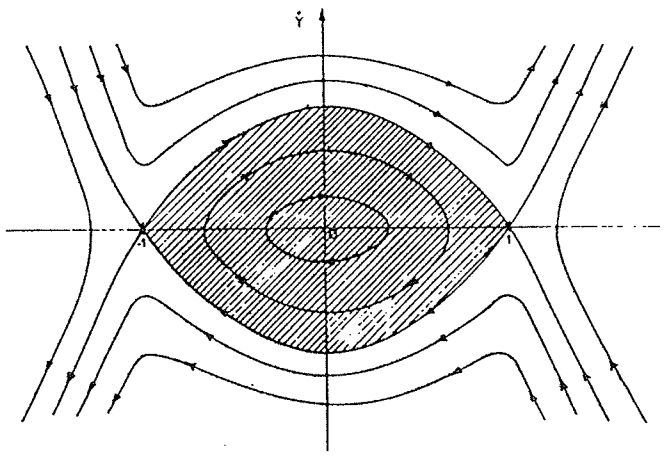
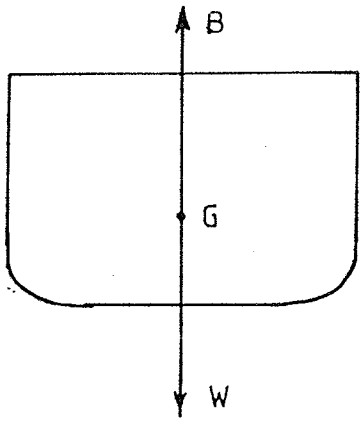
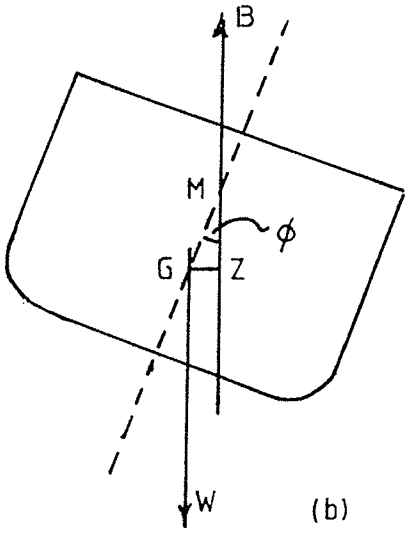


(c)

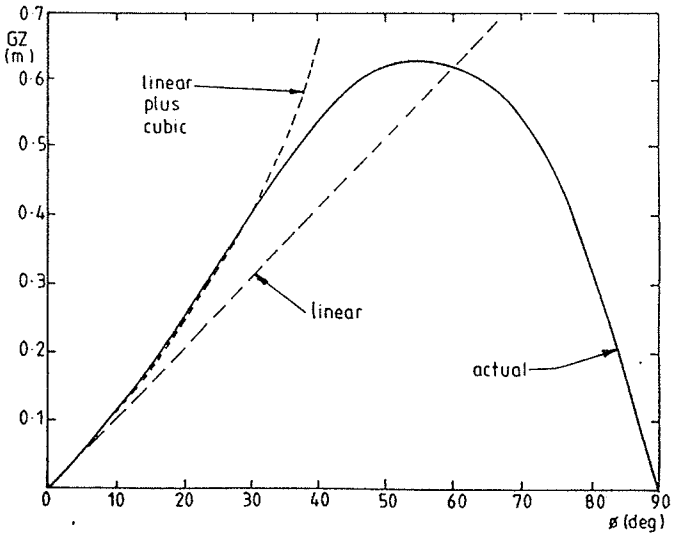
Fig. 1.3.


B
G
W

(a)


B
M
φ
G
Z
W

(b)


GZ (m)
0·7
0·6
0·5
0·4
0·3
0·2
0·1
linear plus cubic
linear
actual
0
10
20
30
40
50
60
70
80
90
ø (deg)

(c)

Fig. 1.4.

3.2 Hydrostatic restoring forces

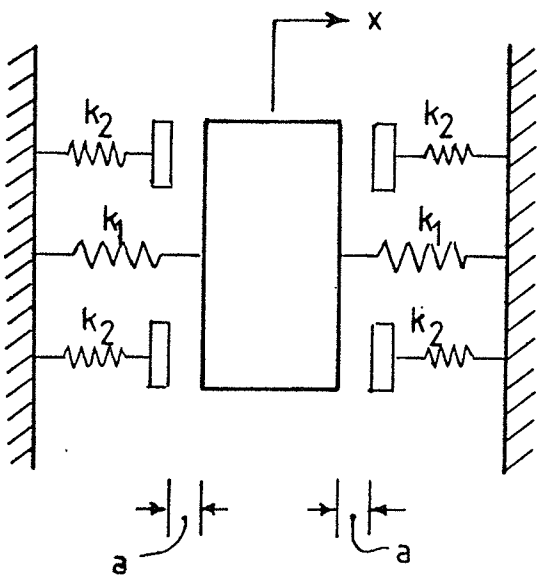
Consider a ship in its position of stable equilibrium, as shown in Fig. 1.4(a). Here the line of action of the weight of the ship, W , coincides with the line of action of the buoyancy force, B .

If the ship is given a displacement, θ , as shown in Fig. 1.4(b), such that the buoyancy force remains constant, then there will now be some distance, r say, between the lines of action of B and W . r is usually called the "righting lever", and for small angles of roll is given by GM $\sin\theta$, where GM is the "metacentric height" - i.e. the distance between the "metacentre", M , and the centre of gravity, G . A typical variation of the righting lever with θ is shown in Fig. 1.4(c). It is seen that there is a critical angle, θ , at which the righting lever, and hence the restoring moment, reduces to zero. This is the position of incipient capsize.

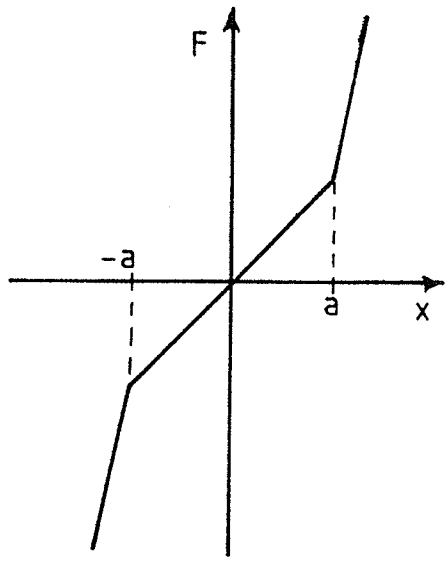
For very small angles of roll a linear approximation for the r-θ characteristic is appropriate. However, if one wishes to study the probability of capsize, using a dynamic model, then it is obviously essential to model the highly non-linear behaviour of the r-θ curve, at high roll amplitudes.

3.3 Geometric non-linearities

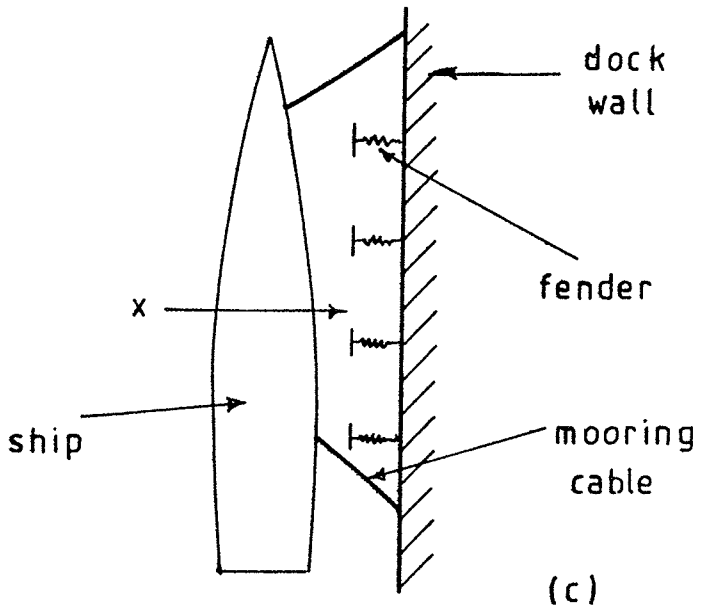
An overall non-linear force-displacement characteristic can occur in situations where linear springs are arranged in certain geometrical ways. An example is shown in Fig. 1.5(a). Here the mass m is restrained by a spring of stiffness k , and if the amplitude of motion is less than 'a' , the combined stiffness is $2k_1$. However, if the amplitude of motion exceeds 'a' , the springs of stiffness k_2 become active and here is an abrupt transition to the overall stiffness of $2k_1 + 4k_2$. Thus, as


x
k2
k2
k1
k1
k2
k2
a
a

(a)


F
-a
a
x

(b)


dock
wall
fender
x
ship
mooring
cable

(c)

Fig. 1.5.

shown in Fig. 1.5(b), the overall restoring force - versus amplitude characteristic is only piece-wise linear.

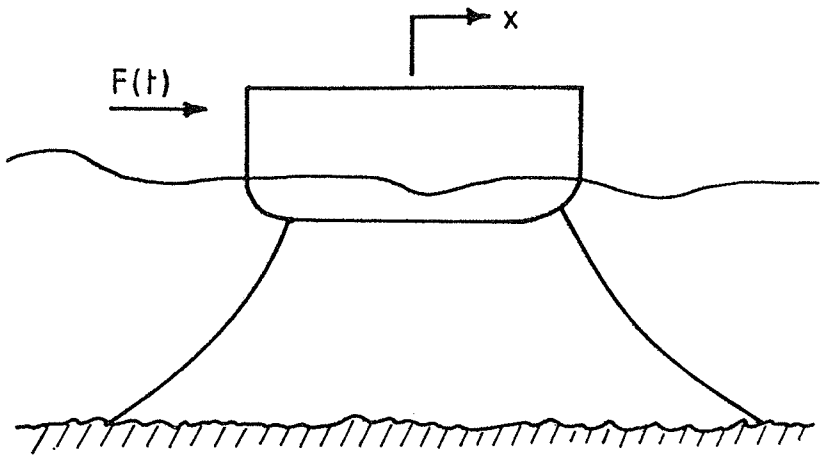
This type of non-linearity occurs whenever "elastic stops" are used in an attempt to limit the amplitude of motion of the mass in a system. A practical example occurs in the situation where a ship is moored to a dock wall by means of cables. To stop the ship impacting the wall, fenders are usually introduced, as indicated in Fig. 1.5(c). Thus for motion in a direction orthogonal to the wall the net restoring force is highly non-linear in character. It has been shown that the motion, due to wave action, of such moored ships can exhibit many of the classical non-linear phenomena - i.e., subharmonic resonances, jumps, etc.

Non-linear restoring forces can also occur in the case of marine vessels, or offshore platforms, moored in the open sea, as indicated in Fig. 1.6(a). Through non-linear interaction between the wave motion and the floating body, the wave force experienced by the body has a very low frequency content, commensurate with the natural frequency of drift motion. Due to the geometry of the mooring cable configuration the restoring force characteristic can be highly non-linear, as indicated in Fig. 1.6(b). If one allows for the ability of the cables to stretch elastically then the degree of non-linearity is somewhat reduced, but often by no means eliminated.

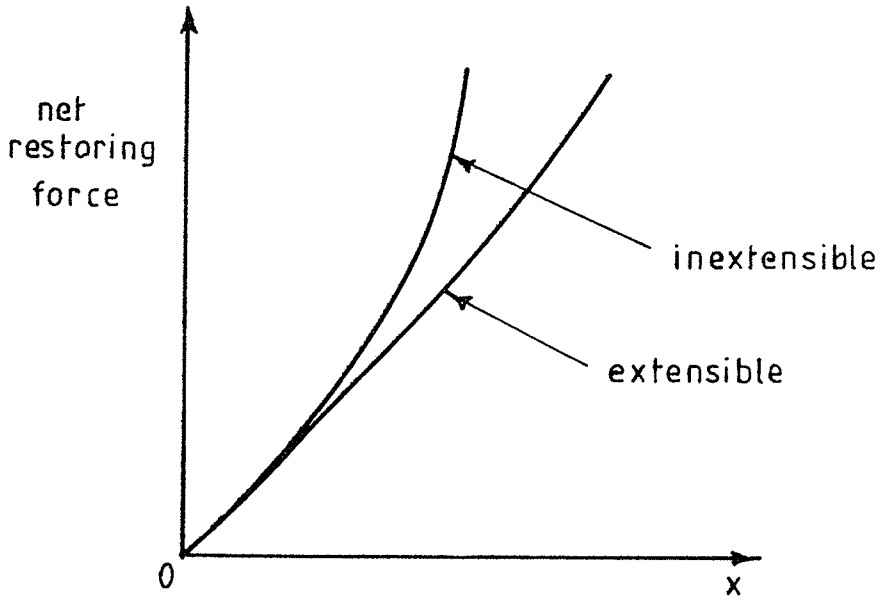
3.4 Strain-displacement non-linearities

Non-linear strain-displacement relationships can often result in overall non-linear restoring force characteristics, even in situations where the material stays within the linear elastic regime (i.e. the constitutive stress-strain law of the material is linear).

A simple example of this occurs when a plate, which is rigidly restrained at its perimeter, is loaded transversely, as indicated in Fig. 1.7(a). Due to the effect of the membrane



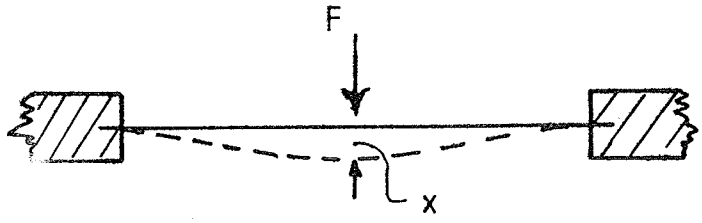
(a)



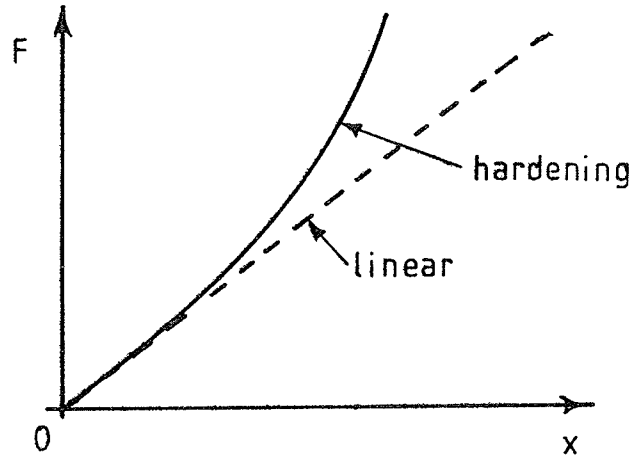
(b)

Fig. 1.6.

forces in the plate, during elastic deformation, the force-displacement characteristic exhibits the "hardening" trend shown in Fig. 1.7(b) (for an analytical treatment, see Lin (1967)).



(a)



(b)

Fig. 1.7.

A very simple model can be used to demonstrate this type of non-linearity. Fig. 1.8(a) shows a mass, m , restrained by two springs, of stiffness k , with an initial tension T_0 . If x is the transverse displacement due to a transverse load, F , then the F-x characteristic is as shown in Fig. 1.8(b). For small displacements the restoring force is approximately linear. However for larger amplitudes of displacement the system exhibits either a hardening, or softening characteristic (depending on the values of T_0 and k) before asymptoting towards linear behaviour again (but with a new stiffness) at very large displacements. This non-linearity occurs even though the individual springs are assumed to behave in a linear fashion.

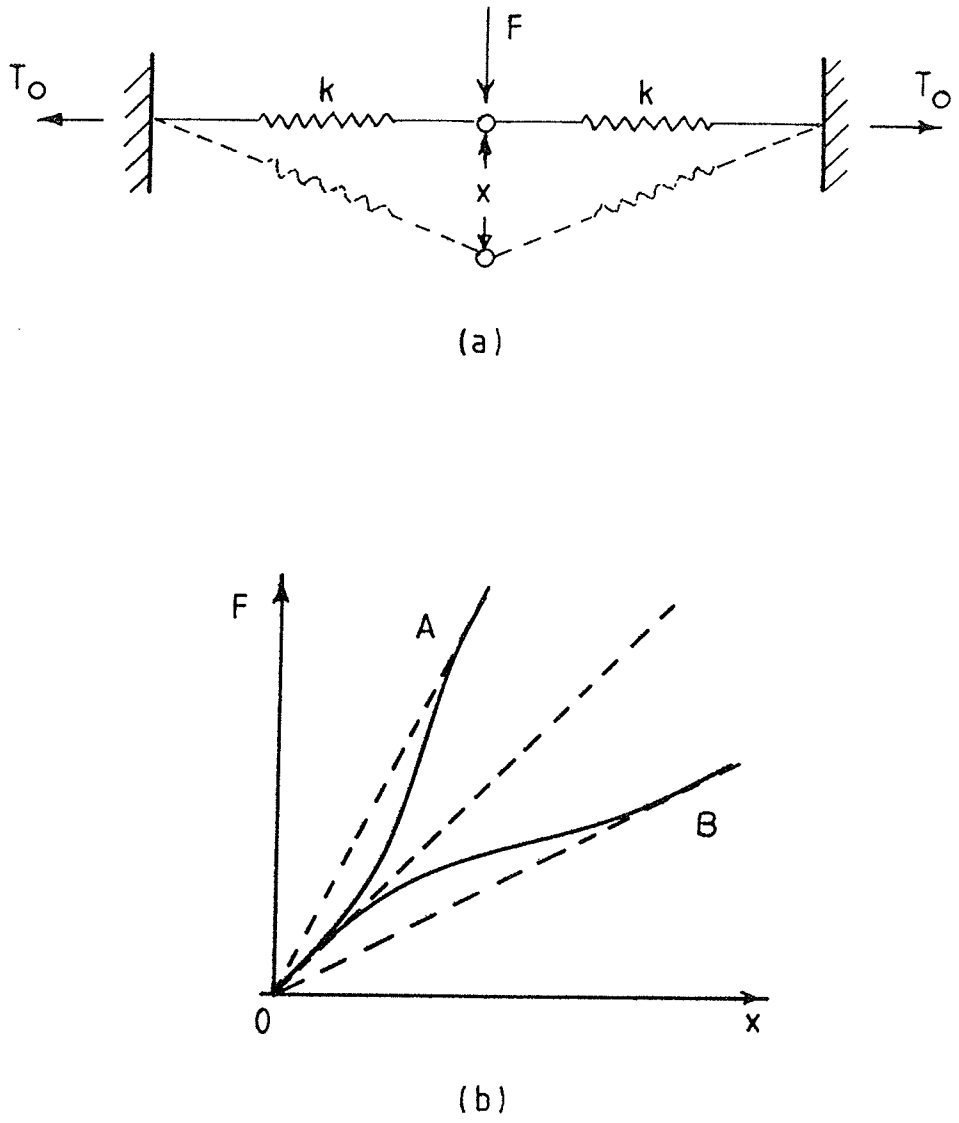
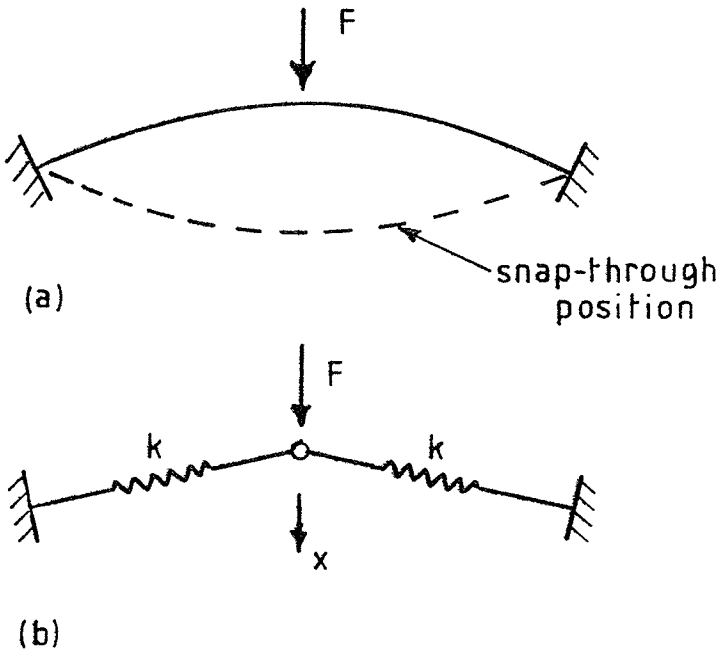


Fig. 1.8.

A slight modification of this simple system can be used to illustrate the non-linear phenomenon of "snap-through", which occurs when shallow shells (see Fig. 1.9(a)) are subjected to transverse loading. At a critical value of the load the shell will "snap-through" to a new position, as indicated in the figure. The simple spring-model shown in Fig. 1.9(b) exhibits this phenomenon. It is a simple matter (e.g. see Thompson & Hunt (1973)) to work out the force-displacement characteristic, as shown in Fig. 1.9(c). The section of the curve labelled BCD can be shown to be unstable. Thus loading in a positive direction

results in the path ABE , where reversal of loading results in the path EDF . Snap-through occurs from B to E , and from D to F .



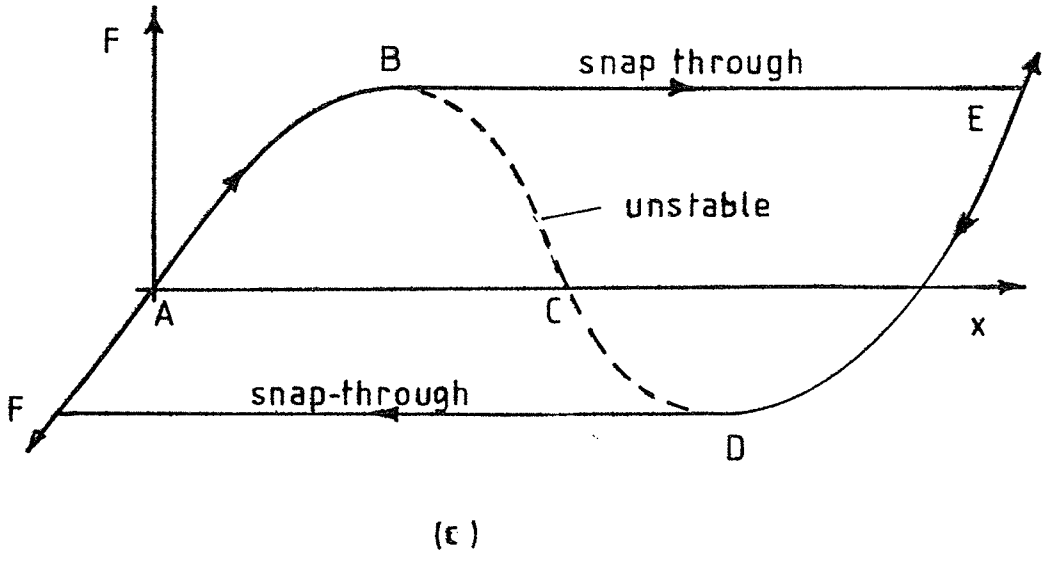


Fig. 1.9.

3.5 Non-linear elasticity

Some materials behave with non-linear stress-strain constitutive laws, but still remain (approximately) conservative. For example, mild steel has the stress-strain behaviour indicated in Fig. 1.10(a). In the small range between the "proportionality" limit, P and the "elastic limit", E , the material behaves non-linearly, but elastically. Thus the same path is taken between E and P , irrespective of the direction of loading. If

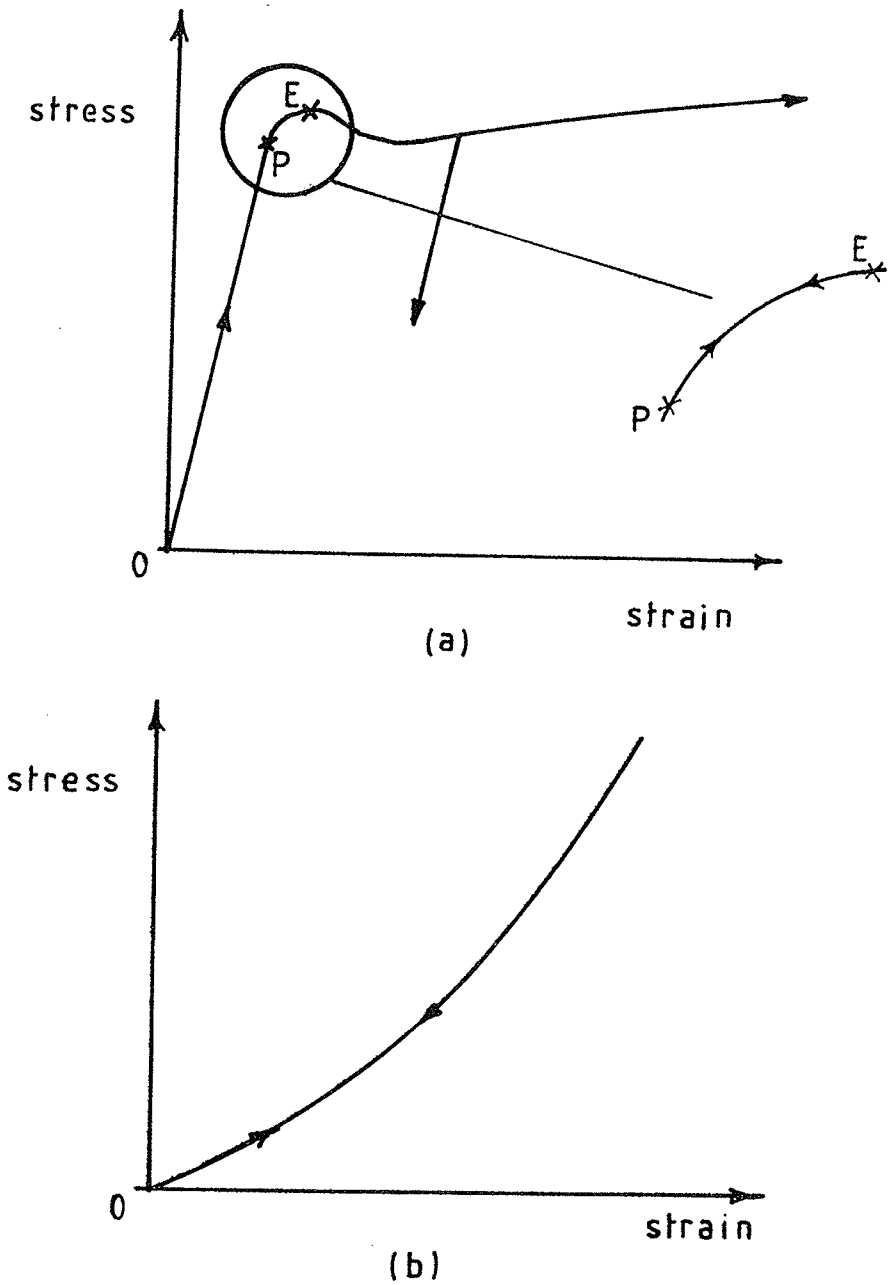


Fig. 1.10.

the loading is increased beyond E , and then reversed, then a new path is followed, as shown in the figure, and, under cyclic loading, one has the phenomenon of hysteresis, with associated dissipation of energy.

Some materials, such as elastomers, exhibit highly non-linear stress-strain curves, as indicated by Fig. 1.10(b). To a rough degree of approximation such materials can be modelled in terms of conservative forces of a non-linear type, although some degree of energy dissipation is always present (as indeed it is in all materials under cyclic loading).

4. Examples of non-linear dissipative forces

Damping forces are often measured experimentally, from the free-decay of single-degree-of-freedom system, due to the inherent difficulty of deriving mathematical models from basic principles, in most applications. The discussion of the representation of damping, given here, will therefore start with the means of extracting suitable information from free-decay tests.

With no external force acting, the general equation of motion of an oscillator with non-linear damping may be written as (assuming that the inertial term is linear)

$$a\ddot{q} + \frac{\partial V}{\partial q} = - Q^D \tag{30}$$

If one defines a total energy function, E , as

$$E = \frac{1}{2} a \dot{q}^2 + V(q) \tag{31}$$

where the first term is the kinetic energy contribution and the second term is the potential energy contribution, then equation (30) can be re-expressed as

$$\frac{dE}{dt} = - L(E) \tag{32}$$

where

$$L(E) = Q^D \dot{q} \tag{33}$$

The function of E, $L(E)$, will be called here the "loss function". Strictly this function depends on time, as well as E, but, as will be shown later, can be treated as a function of E only, to a good approximation, for lightly damped systems.

It is useful to define a non-dimensional damping function, $Q(E)$, associated with $L(E)$, as follows:

$$Q(E) = \frac{L(E)}{2\omega_0 E} \tag{34}$$

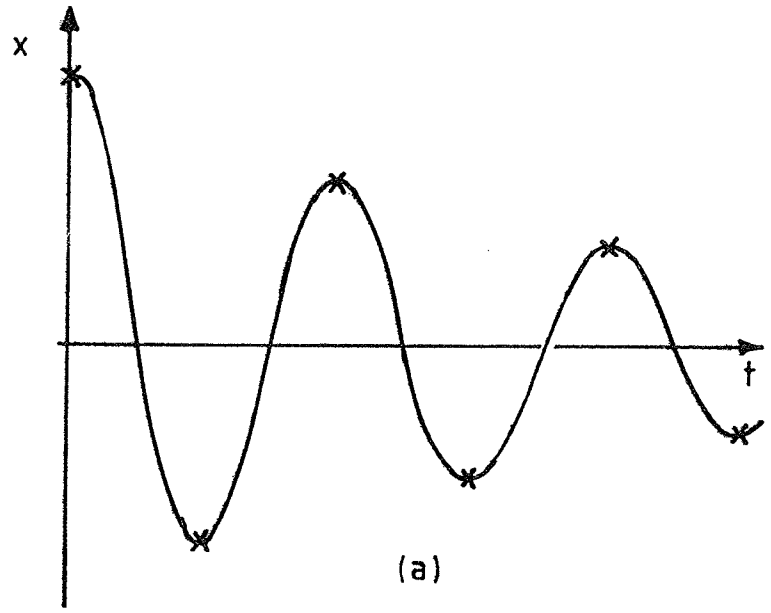
where ω_0 is the frequency of small oscillations. $Q(E)$ is proportional to the fractional loss of energy per cycle due to damping - i.e. the ratio of the energy lost per cycle to the total energy in that cycle.

In the special case of linear damping the damping function $Q(E)$ is independent of amplitude, as measured by E. In fact one finds that

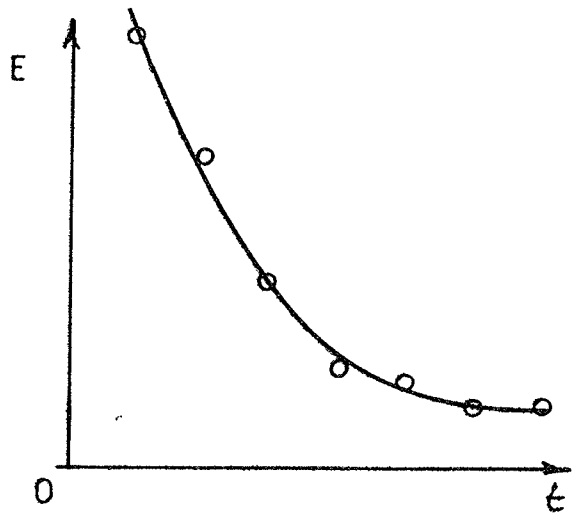
$$Q(E) = \zeta \tag{35}$$

where ζ is the usual non-dimensional damping factor ($\zeta = 1$ for critical damping).

The loss function $L(E)$ and the damping function, $Q(E)$, can be estimated fairly easily from free-decay data. Suppose one has a free-decay record, as illustrated in Fig. 1.11(a). The amplitudes of the peaks, $A_i = x(t_i)$, which occur at times t_i, may be converted into corresponding energy values, E_i, through the relationship (see equation (31))



(a)



(b)

Fig. 1.11.

$$E_i = V(A_i) \tag{36}$$

If E_i is plotted against time, as indicated in Fig. 1.11(b), and a curve fitted through the data, one can form an estimate of $L(E)$ from the gradient of the curve, at various values of E

(see equation (32)). Hence $Q(E)$ can be evaluated, according to equation (34).

It will be shown later that, when applying the method of stochastic averaging, it is not required to have an explicit mathematical model of the dissipative forces. All that is required is a knowledge of fractional energy loss per cycle, as expressed by the function $Q(E)$. Thus experimental information on damping, as expressed by $Q(E)$, can be directly incorporated into the mathematical analysis. This feature represents one of the major advantages of the stochastic averaging method.

If, however, an explicit, parametric representation of damping is required then this can be derived fairly readily. Suppose, for example, that one assumes that the damping force, Q^D, can be represented by

$$Q^D = \alpha\dot{x} + \beta\dot{x}^3 \qquad (q = x) \tag{37}$$

One can evaluate $L(E)$ from the use of equations (33) and (37). Thus

$$L(E) = (\alpha\dot{q} + \beta\dot{q}^3)\dot{q} \tag{38}$$

Now, if the damping is light, the oscillatory behaviour of the right-hand side of equation (38) can be "ironed-out" by treating the energy level as constant, during one "cycle", provided that the damping is light (this is an extended version of the Krylov-Bogoliubov averaging method). Thus one can write

$$L(E) = \frac{1}{T(E)} \int_0^{T(E)} (\alpha\dot{q} + \beta\dot{q}^3)\dot{q}\, dt \tag{39}$$

where $T(E)$ is the period of undamped, free oscillations, as given by

$$T(E) = 4 \int_0^b \frac{dx}{[2(E-V(x))]^{\frac{1}{2}}} \qquad (40)$$

where $V(b) = E$. On evaluating the integrals one finds that $L(E)$ can be expressed as

$$L(E) = \alpha A(E) + \beta B(E) \qquad (41)$$

where $A(E)$ and $B(E)$ are known functions. Equation (41) can now be matched to an experimentally determined $L(E)$ versus E characteristic, in some optimum way (e.g. least squares) to yield estimates of the parameters α and β .

4.1 Material damping

A wide variety of mechanisms, at the microscopic level, have been identified (e.g. Lazan (1968)) which contribute to energy dissipation in materials (both uniform and composite) under cyclic loading. However, current information is not usually sufficient to allow one to predict energy loss functions quantitatively, for basic physical principles, except in certain very special cases.

It is usual to rely on experimental measurements of the loss function, as pointed out earlier. An example is the study of Roberts and Yousri (1978), where $Q(E)$ was determined for two small cantilevered beams. Fig. 1.12 gives a sketch of the results. For the beam made of steel the function $Q(E)$ was found to be sensibly independent of amplitude, suggesting that a linear model of damping was adequate. However, for the other beam, made of a copper-alloy, $Q(E)$ was found to increase sharply with amplitude, when the latter exceeded a critical level. This is strong evidence of non-linearity, and is typical of many materials loaded cyclically at high amplitudes.

Energy loss functions can be cast into a more generally useful form by introducing the "specific damping function", D_s .

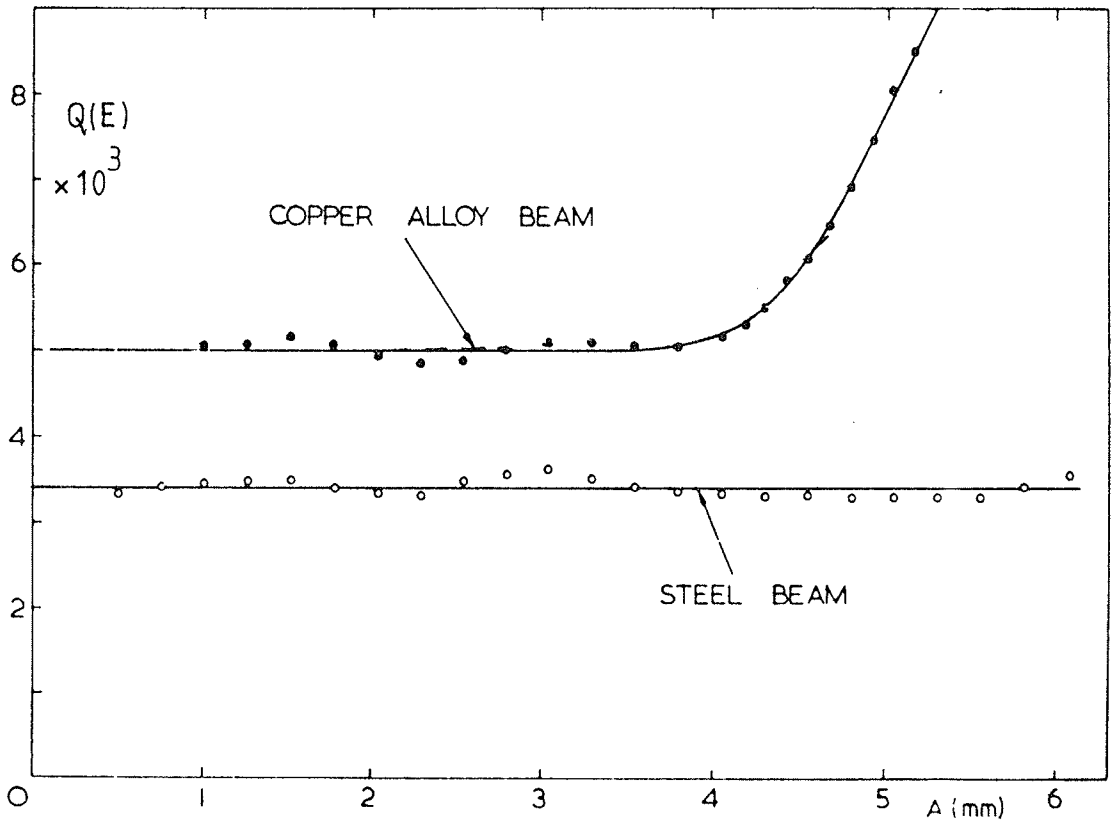


Fig. 1.12.

This is defined as the energy loss per unit volume of material per cycle. D_s can be easily related to L(E) , if the mode of vibration is known, because the instantaneous stress amplitude distribution is then known. Since D_s is a function of stress amplitude it is a simple matter to convert a stress distribution into a distribution of D_s . One can then obtain L(E) by integrating over the volume of the structure.

Fig. 1.13 sketches a typical variation of D_s with σ/σ_e , where σ is the stress amplitude and σ_e is a reference stress, defined as the fatigue strength of the material at 2×10^7 cycles. The band shows the spread covered by about 20 different materials. At low amplitudes D_s varies (approximately) with σ/σ_e in a quadratic fashion, as indicated by the mean line. This is the result obtained from a simple linear viscous model, suggesting that this type of modelling is adequate at low vibra-

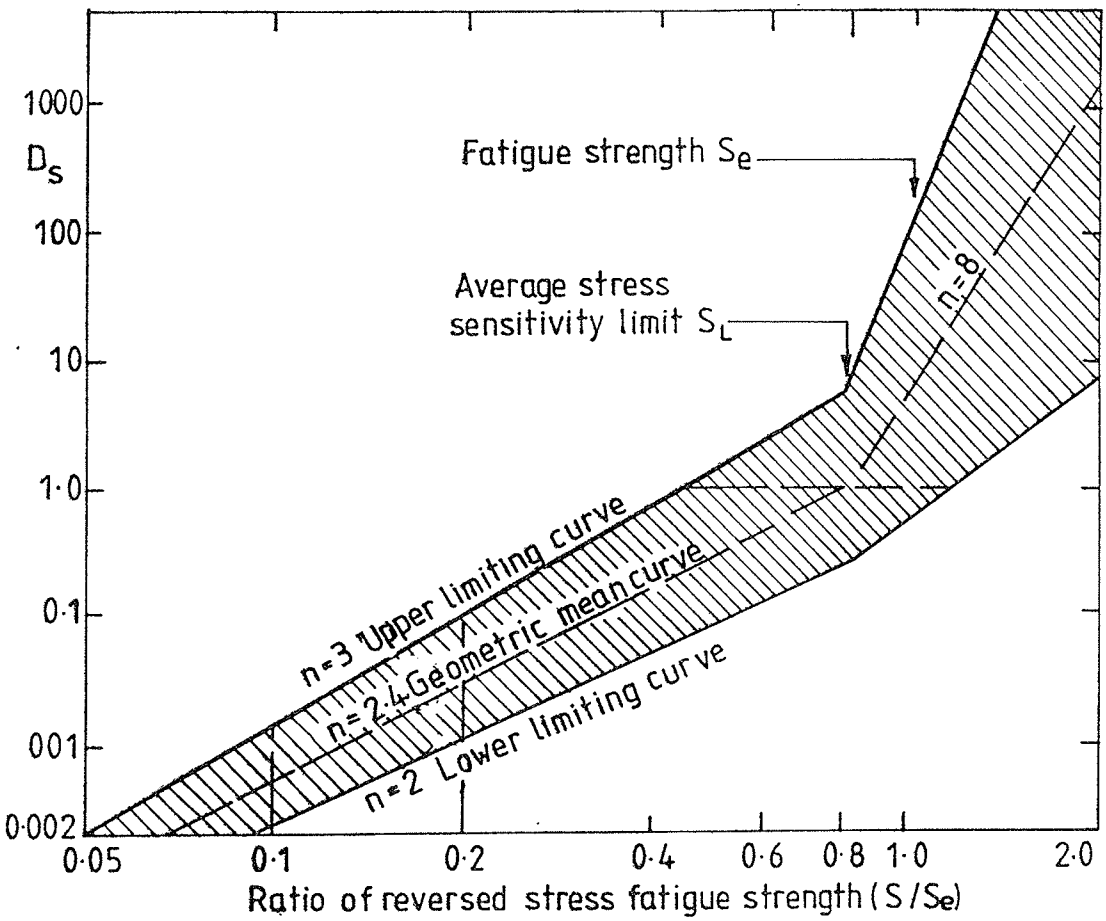


Fig. 1.13.

tion amplitudes. At a value of $\sigma/\sigma_e \sim 1$ there is sharp break in the variation of D_s with σ/σ_e, and for higher stress D_s increases very rapidly with stress level. In this regime the fatigue life is fairly short. The mean variation of D_s with σ/σ_e is represented reasonably well by the relationship (Lazan (1968))

$$D_s = \alpha\left[\left[\frac{\sigma}{\sigma_e}\right]^2 + \beta\left[\frac{\sigma}{\sigma_e}\right]^n\right] \qquad (42)$$

where α, β and n are material constants and $n >> 2$. The second term of the above represents the non-linear contribution to damping.

4.2 Approximate modelling of material forces

The total force resulting from material deformation can be modelled, approximately, as the sum of

(a) a conservative restoring force (with an associated potential energy function,
(b) a zero memory velocity dependent dissipative force.

Thus

$$Q = Q^D + Q^R \tag{43}$$

where

Q^D - dissipative

Q^R - conservative

A simple parametric form for Q^D may be chosen - e.g.

$$Q^D = \alpha\dot{q} + \beta\dot{q}^n \tag{44}$$

The parameters α , β and n can be found by matching the energy loss function resulting from equations (43) and (44) to the experimentally determined specific loss function, D_s , variation with amplitude.

This non-hysteretic approach to modelling is usually sufficiently accurate, for practical purposes, if the damping is very light.

4.3 Coulomb damping

Coulomb type damping occurs whenever there is dry, (i.e unlubricated) contact between two surfaces in relative, sliding motion. Thus, as in Fig. 1.14(a), the movement of a mass, m , on a rigid horizontal surface, with velocity v , is resisted by a friction μmg , where μ is the coefficient of friction. For idealised Coulomb damping the frictional force is independent of

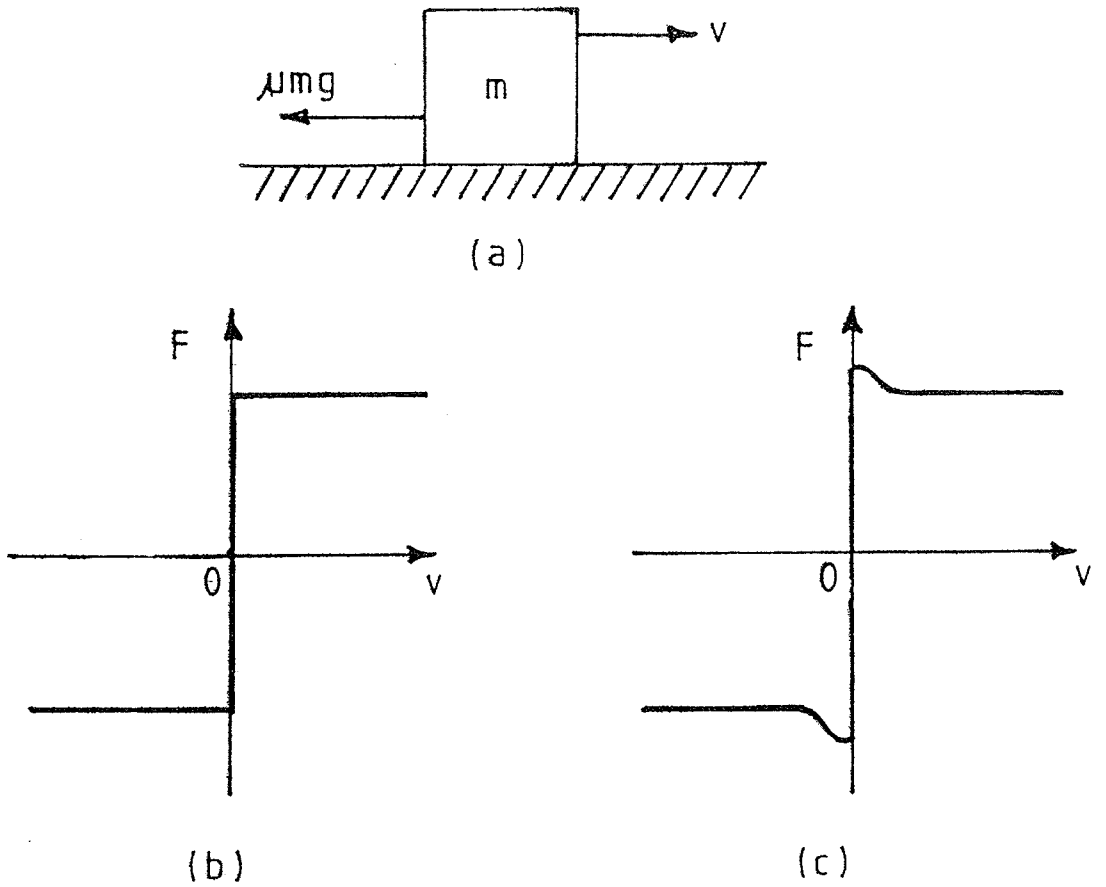


Fig. 1.14.

the magnitude of the velocity, and hence has the characteristic shown in Fig. 1.14(b). Actual friction characteristics are rather smoother, and there is a "hump", near $|v| = 0$, associated with "static friction". Once movement commences the frictional force tends to drop slightly - i.e. dynamic friction is usually a little less than static friction. Various "improved" models of dry friction have been proposed, and used in the modelling of drill strings occurring in oil wells.

4.4 Hysteretic damping

As indicated earlier, the restoring force resulting from deforming materials is never exactly conservative. If a material

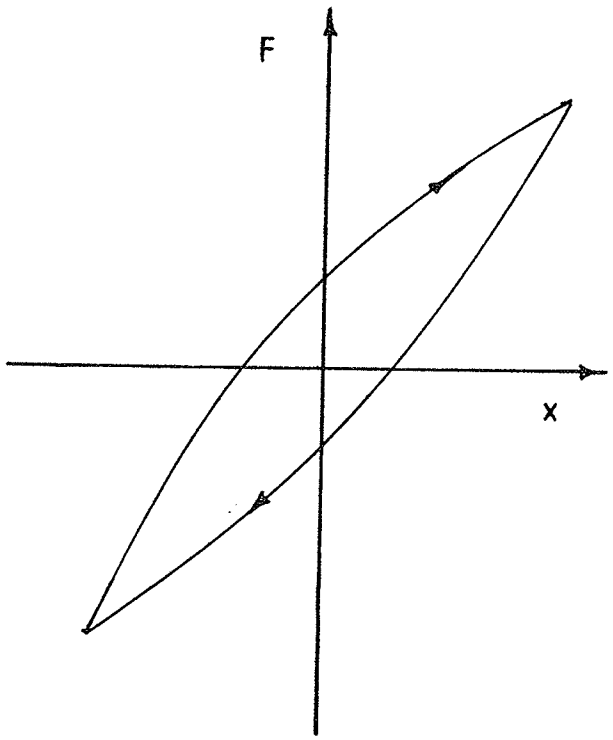


Fig. 1.15.

is subjected to cyclic loading then, during one cycle, the load displacement characteristic will appear as indicated in Fig. 1.15. Due to the hysteretic nature of the material there is "loop" in the characteristic and it is easy to see that the energy dissipated per cycle is directly proportional to the area of the loop.

Such loops exist even in situations where the material is stressed within its elastic limit, and the stress-strain law is very closely modelled by straight line (as in the case, for example, of mild steel). Of course, in this case the dissipative force are, relatively, very small; hence the loops are very slim, and may be difficult to detect experimentally.

At the other extreme, when materials are loaded well beyond their elastic limit, or there is some relative sliding between

the components of composite structures, the energy dissipation can be, relatively, very large and the loops become very "fat".

There are two basic types of hysteresis loop, as sketched in Fig. 1.16. Elliptical loops are associated with linear damping mechanisms and are characterised by the fact that the energy loss per cycle is proportional to the square of the amplitude of oscillation, A . Non-elliptical loops can occur in a wide variety of shapes but usually have sharp corners, as indicated in the figure. For such loops the energy loss is not proportional to the amplitude of oscillation squared and the underlying dissipative mechanism is non-linear in nature.

Recently there has been considerable interest in the representation of hysteretic loops by means of differential equations (although such an approach does, in fact, have quite a long history (Lazan (1968))).

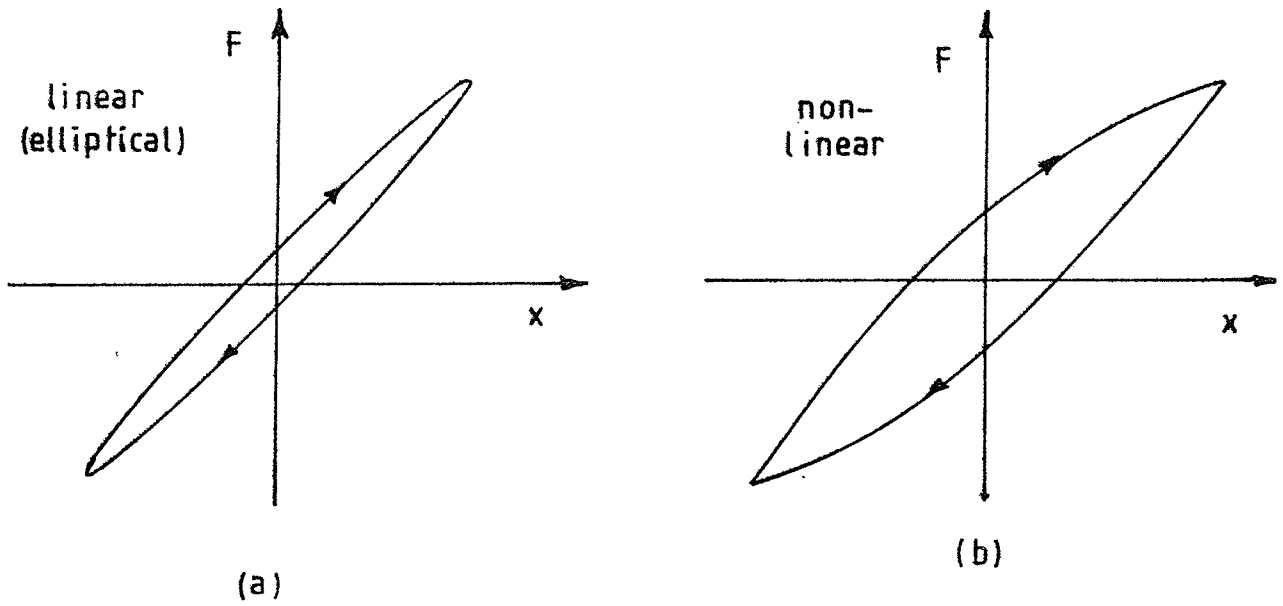


Fig. 1.16.

Elliptical loops can be modelled very simply as the sum of a linear restoring force and a viscous, linear damping term - i.e. -

$$F(t) = ax + b\dot{x} \qquad (44)$$

where a and b are constants. The difficulty with this representation, however, is that the energy loss per cycle is frequency dependent, whereas actual energy loss functions are usually relatively insensitive to the frequency of excitation. However, if one is dealing with a system with a single, dominant resonance then one is primarily concerned with energy dissipation at the resonant frequency, and a and b in the above can be chosen to give the correct energy loss at this frequency. Errors in representation at other frequencies are then relatively insignificant.

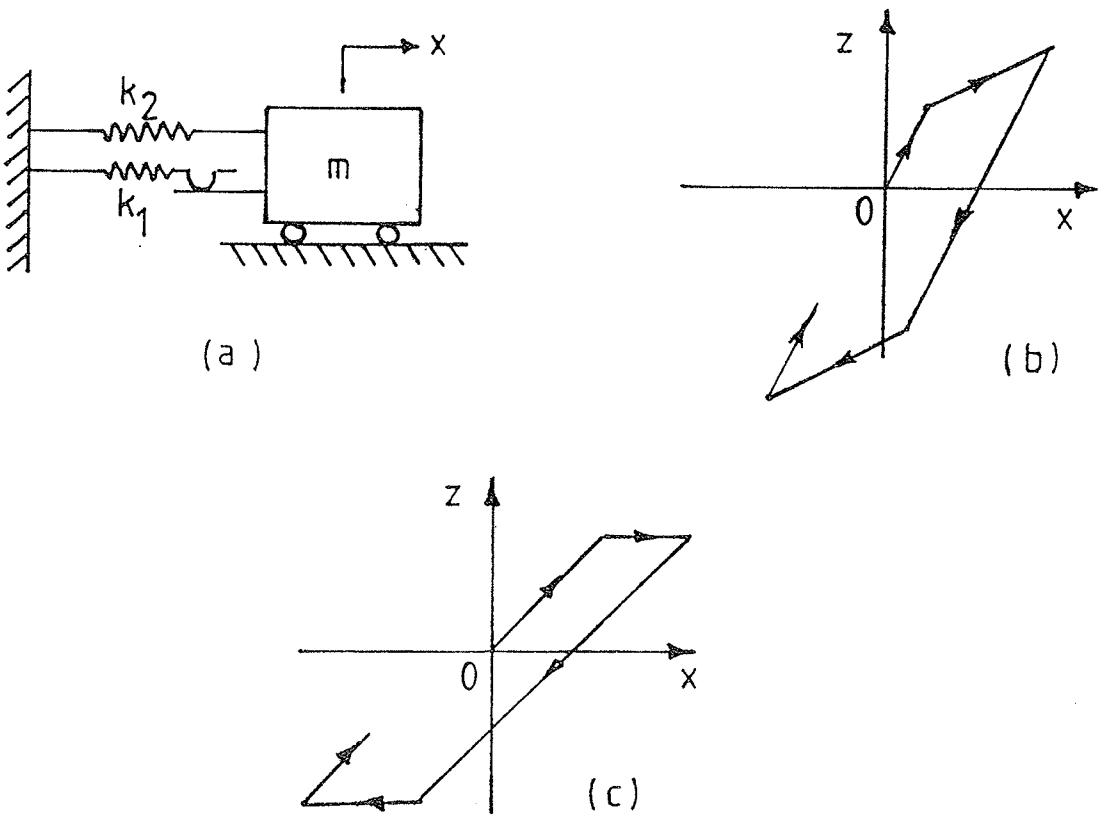


Fig. 1.17.

An example of the occurrence of a non-elliptional loop is furnished by the mass-spring system shown in Fig. 1.17(a). Here the spring k, is in series with a Coulomb damping element. If the mass undergoes harmonic motion, of sufficient amplitude then the force-displacement characteristic is of the "bilinear hysteretic" kind shown in Fig. 1.17(b). This is piece-wise linear and there are two slopes of relevance - that corresponding to no sliding and that corresponding to sliding motion in the Coulomb damping element. A special case of such hysteresis is "elasto-plastic" hysteresis, where the secondary slope is zero, as shown in Fig. 1.17(c).

A general differential form for a hysteretic force, z, is

$$\dot{z} = G(\dot{x},z) \tag{45}$$

where z is the force and x is the displacement. As a specific example, if the hysteresis is elasto-plastic one has

$$G(\dot{x},z) = \dot{x}[1 - U(\dot{x})\ U(z-1) - U(-\dot{x})\ U(-z-1)] \tag{46}$$

where $U(\ )$ is the unit step function.

Equation (43) may be combined with the differential equation of motion for the system - e.g.

$$\ddot{x} + \beta\dot{x} + \alpha x + (1-\alpha)z = f(t) \tag{47}$$

where α is a constant, would be appropriate for an oscillator with some degree of hysteresis. Combining equations (45) and (47) one has a third-order system which can be treated by normal, "non-hysteretic" methods.

Other forms of $G(\dot{x},z)$ are possible. For example, one has the Bouc-Wen model (Bouc (1967), Wen (1980))

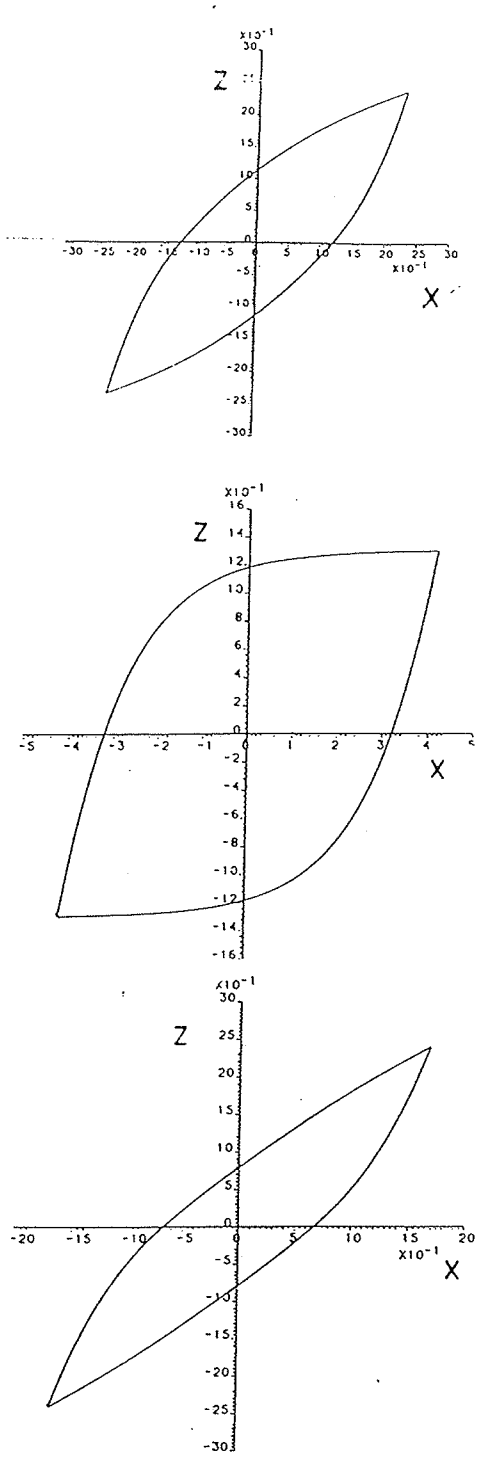


Fig. 1.18.

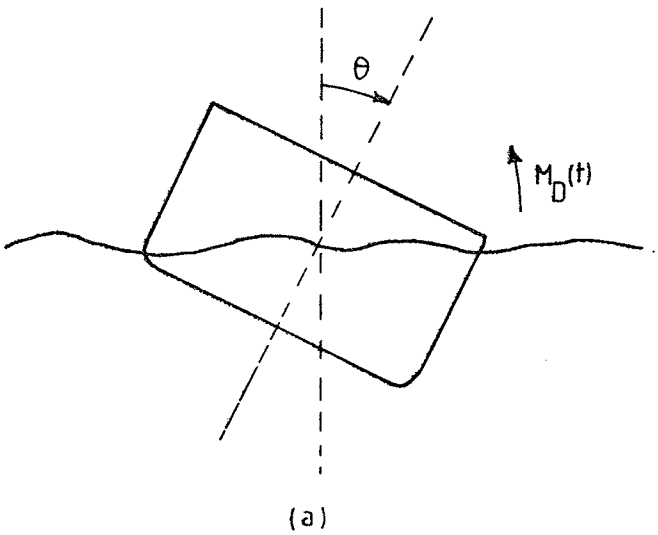
$$G(\dot{x},z) = -\gamma|\dot{x}|z|z|^{n-1} - \nu\dot{x}|z|^{n} + Ax \qquad (48)$$

where γ, ν, A and n are "loop parameters", which control the shape and magnitude of the loop. Some typical shapes of loop which can be represented by equation (48) are sketched in Fig. 1.18.

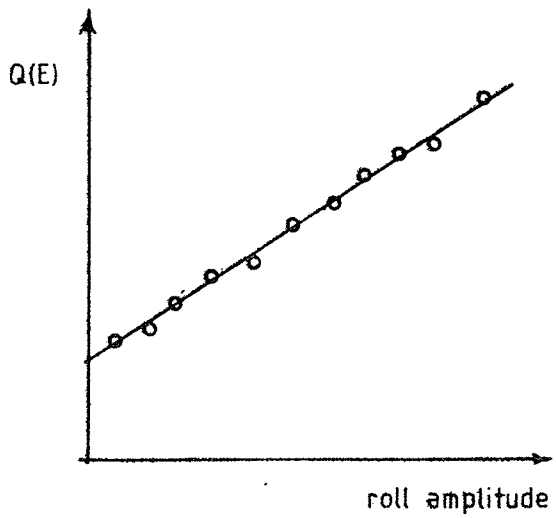
4.5 Fluid-Structure interaction

Many engineering problems involve the dynamic interaction between moving objects and a surrounding fluid.

As an example, consider a ship rolling in waves, as sketched in Fig. 1.19(a). It can be shown by experimental studies dating back over one hundred years to the pioneering work of Froude (see Froude (1955)) that the dissipative moment, $M_D(t)$, during roll motion is well represented by a linear-plus-quadratic form. Thus



(a)



(b)

Fig. 1.19.

$$M_D(t) = \alpha\dot{\theta} + \beta|\dot{\theta}|\dot{\theta} \qquad (49)$$

where α and β are constants. The second, quadratic term here arises from flow separation effects, with associated vortex shedding. The typical variation of the damping function, Q(E) , with roll amplitude is shown in Fig. 1.19(b). This shows that the linear contribution, as given by the intersect on the vertical axis, is relatively small and at moderate angles of roll the principal contribution to damping is from the quadratic term.

Quadratic damping occurs frequently in problems involving fluid-structure interaction. As another example, consider the case of the fluid forces acting a vertical cylinder, such as occur in offshore oil production platforms. Numerous workers have shown that the fluid force, f(t) , on a cylinder moving with velocity v , is well represented by Morison's equation (Morison (1950)). This is given by

$$f(t) = \rho A\ddot{v} + C_I\rho A(\ddot{v}-\ddot{x}) + \frac{1}{2} C_D\rho D|\dot{v}-\dot{x}|(\dot{v}-\dot{x}) \qquad (50)$$

where

ρ = fluid density
A = cross-sectional area, normal to the flow
D = diameter of cylinder
C_I = added mass coefficient
C_D = drag coefficient

The first two terms in the above expression are linear, inertial terms. The last term is a quadratic, drag component, and is non-linear.

5. Analytical methods

We now review, briefly, the various analytical approaches which are available for solving non-linear random vibration problems.

5.1 Markov methods

The general equations of motion of an n degree of freedom system, as given by equation (26) are conveniently written in state space form. Introducing the state variable vector

$$\underset{\sim}{z} = \begin{bmatrix} \underset{\sim}{q} \\ \dot{\underset{\sim}{q}} \end{bmatrix} \tag{51}$$

one can rewrite equation (26) as

$$\dot{\underset{\sim}{z}} = \underset{\sim}{a}(\underset{\sim}{z}) + \underset{\sim}{F}\cdot\underset{\sim}{Y}(t) \tag{52}$$

where

$$\underset{\sim}{a}(\underset{\sim}{z}) = \begin{bmatrix} \dot{\underset{\sim}{q}} \\ -\underset{\sim}{A}^{-1}\underset{\sim}{B}\dot{\underset{\sim}{q}} - \underset{\sim}{A}^{-1}\underset{\sim}{K}\underset{\sim}{q} - \underset{\sim}{A}^{-1}\underset{\sim}{\Phi} \end{bmatrix} \tag{53}$$

and

$$\underset{\sim}{F} = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{A}^{-1} \end{bmatrix} \qquad \underset{\sim}{Y} = \begin{bmatrix} \underset{\sim}{0} \\ \underset{\sim}{Q} \end{bmatrix} \tag{54}$$

Note that it will be assumed here that $\underset{\sim}{F}$ can not contain terms dependent on $\underset{\sim}{z}$ - i.e. parametric excitation is absent. For a discussion of parametric excitation effects, see Ibrahim (1985).

If the elements of $\underset{\sim}{Y}(t)$ are broad-band in character they can, in many cases, be satisfactorily approximated in terms of white noises. Thus

$$\underset{\sim}{Y}(t) = \underset{\sim}{K}\,\underset{\sim}{\xi}(t) \tag{55}$$

where $\underset{\sim}{K}$ is a 2n×2n matrix. $\underset{\sim}{\xi}(t)$ is a 2n-vector of independent, unit white noises, with a correlation matrix

$$E\{\underset{\sim}{\xi}(t)\ \underset{\sim}{\xi}^{T}(t+\tau)\} = \underset{\sim}{I}\,\delta(\tau) \tag{56}$$

where $\underset{\sim}{I}$ is a 2n×2n unit matrix.

Equation (52) can now be written as

$$\dot{\underset{\sim}{z}} = \underset{\sim}{a}(\underset{\sim}{z}) + \underset{\sim}{B}\,\underset{\sim}{\xi}(t) \tag{57}$$

where

$$\underset{\sim}{B} = \underset{\sim}{F}\,\underset{\sim}{K} \tag{58}$$

Equation (57) must be carefully interpreted. For example, $\dot{\underset{\sim}{z}}$ exists almost nowhere (see, for example, Sobczyk (1985)). One suitable interpretation, and that adopted by many authors, is to treat equation (57) as an Itô equation; it may be written in standard form as (Sobczyk (1985))

$$d\underset{\sim}{z} = \underset{\sim}{a}(\underset{\sim}{z})dt + \underset{\sim}{B}\,d\underset{\sim}{W} \tag{59}$$

where $\underset{\sim}{W}$ is a 2n-vector of unit "Wiener processes", such that (formally)

$$\underset{\sim}{\xi}(t) = \frac{d\underset{\sim}{W}}{dt} \tag{60}$$

When the excitation is modelled in terms of white noises it follows that $\underset{\sim}{z}$ is a 2n-dimensional Markov process, with a transition density function, $p(z|z_0;t)$ governed by the following diffusion equation (see Sobczyk (1985)):

$$\frac{\partial p}{\partial t} = Lp \tag{61}$$

where

$$L = -\sum_{i=1}^{2n} \frac{\partial}{\partial z_i}\left[a_i(\underset{\sim}{z})\cdot\right] + \frac{1}{2}\sum_{i,j=1}^{2n}\sum^{2n} D_{ij}\,\frac{\partial^2}{\partial z_i \partial z_j} \tag{62}$$

and

$$\underset{\sim}{D} = [D_{ij}] = \underset{\sim}{B}\,\underset{\sim}{B}^T \tag{63}$$

Equation (61) is generally known as the Fokker-Planck-Kolmogorov (FPK) equation - sometimes it is known alternatively as the "forward Kolmogorov" equation.

The initial condition for $p(\underset{\sim}{z}|\underset{\sim}{z}_0;t)$ is usually of the form

$$\lim_{t\to 0} p(\underset{\sim}{z}|\underset{\sim}{z}_0;t) = \delta(\underset{\sim}{z}-\underset{\sim}{z}_0) \tag{64}$$

This simply states that one knows the position of the system ($\underset{\sim}{z} = \underset{\sim}{z}_0$) at time $t = 0$ - a deterministic start condition.

If the system is stable and time-invariant a "stationary" solution to the FPK equation usually exists, provided that the white noises do not have time-varying strengths - this is implicit in the analysis so far, (see equation (56)). Thus, as time elapses, $p(\underset{\sim}{z}|\underset{\sim}{z}_0;t)$ becomes independent of the initial condition and asymptotes towards a stationary density function, $w(\underset{\sim}{z})$; i.e.

$$w(\underset{\sim}{z}) = \lim_{t\to\infty} p(\underset{\sim}{z}|\underset{\sim}{z}_0;t) \tag{65}$$

$w(\underset{\sim}{z})$ may be obtained as the solution of equation (61) with $\partial p/\partial t = 0$. Thus

$$Lw = 0 \tag{66}$$

A general closed-form solution to equation (61) has yet to be found, for an arbitrary value of n. However, a series solution can be found in terms of eigenfunctions. If the operator L has a discrete set of eigenfunctions, $v_j(\underset{\sim}{z})$, and corresponding eigenvalues, λ_i, that satisfy

$$(L + \lambda_i)v_i = 0 \tag{67}$$

then the transition density function is given by

$$p(\underset{\sim}{z}|\underset{\sim}{z}_0;t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} v_i(\underset{\sim}{z})v_i^*(\underset{\sim}{z}_0) \tag{68}$$

The set of eigenfunctions, $v_i^*(\underset{\sim}{z})$, relate to the adjoint of L and are orthogonal to the set $v_i(\underset{\sim}{z})$; thus

$$\int_{-\infty}^{\infty} v_i(\underset{\sim}{z})\ v_j^*(\underset{\sim}{z})d\underset{\sim}{z} = \delta_{ij} \tag{69}$$

where δ_{ij} is the Kronecker delta.

In the above $\lambda_0 = 0$ and $v_0(\underset{\sim}{z}) = w(\underset{\sim}{z})$, the stationary solution $(v_0^*(\underset{\sim}{z}) = 1)$. In some applications the spectrum of eigenvalues is not completely discrete, as assumed above. There is a continuous portion and equation (68) must be modified accordingly (Atkinson (1973)).

When $m = 2n = 1$ the system is of first-order and the series expansion solution

$$p(z|z_0;t) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \frac{v_i(z)v_i(z_0)}{v_0(z_0)} \tag{70}$$

Analytical expressions for $v_i(z)$ and λ_i have been found in only a few special cases (Atkinson (1973), Karlin and McGregor

(1960), Caughey and Dienes (1961), Atkinson and Caughey (1968), Wong and Thomas (1962), Payne (1968)). However, a general expression for the stationary density function can be easily found from equation (66), where

$$L = -\frac{\partial}{\partial z}[a(z)\cdot] + \frac{B^2}{2}\frac{\partial^2}{\partial z^2} \tag{71}$$

The result is (e.g. see Stratonovich (1967))

$$w(z) = \frac{c}{B^2}\exp\left[\frac{2}{B^2}\int_0^z a(\xi)d\xi\right] \tag{72}$$

where c is a normalisation constant, chosen so that

$$\int_{-\infty}^{\infty} w(z)dz = 1 \tag{73}$$

When $n = 1$ one has a second order system, or oscillator. A general form of equation of motion for such oscillators is

$$\ddot{x} + g(x,\dot{x}) = K\xi(t) \tag{74}$$

where $g(\ )$ is an arbitrary, non-linear function of the displacement, x, and velocity, $\dot{x}$. K is a constant.

Equation (74) can be cast into the standard state-variable form, given by equation (57), if

$$\underset{\sim}{z} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \underset{\sim}{a}(\underset{\sim}{z}) = \begin{bmatrix} z_2 \\ -g(z_1,z_2) \end{bmatrix} \tag{75}$$

and

$$\underset{\sim}{Y}(t) = \begin{bmatrix} 0 \\ K\xi(t) \end{bmatrix} \tag{76}$$

and $\underset{\sim}{F}$ is a unit matrix. Hence, from equations (61) and (62) one finds that the appropriate FPK equation is

$$\frac{\partial p}{\partial t} = - z_2 \frac{\partial p}{\partial z_1} + \frac{\partial}{\partial z_2} (gp) + \frac{K^2}{2} \frac{\partial^2 p}{\partial z_2^2} \tag{77}$$

A complete, exact solution to this equation has been found only for the linear case.

If, however, attention is restricted to the stationary density function, $w(\underset{\sim}{z})$, then there is a well-known result for the case of linear damping and arbitrary non-linear stiffness. If

$$g = \beta z_2 + h(z_1) \tag{78}$$

then the stationary solution may be written as

$$w(\underset{\sim}{z}) = c \exp(-\gamma E) \tag{79}$$

where

$$\gamma = \frac{2\beta}{K^2} \tag{80}$$

and c is another normalisation constant. E is the total energy oscillator - i.e.,

$$E = \frac{z_2^2}{2} + \int_0^{z_1} h(\xi) d\xi \tag{81}$$

Equation (81) has been obtained independently by many authors (Andronov et al. (1933), Kramers (1940), Oliver and Wu (1958), Chuang and Kazada (1959)).

For non-linear damping in oscillators, exact solutions exist only in certain very special cases. If

$$g = \beta z_2 H(E) + h(z_1) \tag{82}$$

then it can be shown that (Caughey (1964))

$$w(\underset{\sim}{z}) = c \exp\{-\gamma \int_0^E H(\xi)d\xi\} \tag{83}$$

This is clearly a generalisation of equation (79).

For higher order systems very few exact solutions are available. It is possible to generalise the result given by equation (83) to apply to a certain very specific class of multi-degree of freedom system, but the applicability of this result is very limited (see Fuller (1969) for an interesting discussion).

In view of the paucity of exact solutions there have been numerous attempts to develop approximate analytical methods, and numerical methods combined with analytical methods. These include iterative methods (Mayfield (1973)) series expansion methods (Atkinson (1973), Stratonovich (1964), Bhandari and Sherrer (1968), Wen (1973)), the use of random walk analogies (Toland and Yang (1971), Roberts (1978,1986)) finite element methods (Langley (1985), Bergman and Spencer (1985)) and the application of path-integral methods (Wehner and Wolfer (1983), Kapitanik (1985)).

An important class of approximate methods goes under the title of stochastic averaging methods, and will form the basis for much of the discussion in the remaining text. At this point therefore, it is sufficient to point out that this powerful methodology, due originally to Stratonovich (1964), is applicable when the response processes of interest are narrow-band in nature and the excitation processes are broad-band. The approach can be regarded as a generalisation of the deterministic averaging procedure due to Bogoliubov and Mitropolsky (see Roberts and Spanos (1986)).

Advantages of the stochastic averaging method may be summarised as follows:

(a) the dimension of the FPK equation is reduced (generally by a factor of two); this considerably simplifies the ensuing analysis,

(b) non-linear damping can be handled very easily, and effectively,

(c) effects of parametric excitation are easily incorporated - stability criteria can be derived in such cases,

(d) exact closed form solutions for the reduced FPK equations are, in many cases, obtainable (at least in the case of stationary response).

5.2 Equivalent linearisation methods

Because linear systems are so much easier to analyse than non-linear ones, a natural approach to attacking non-linear problems is to replace a given set of non-linear equations by an equivalent set of linear ones; the difference between the sets is minimised, in some sense (usually a least-square sense).

The method originated in the work of Booton (1954) and Caughey (1963). For reviews of recent developments see Roberts (1981, 1984), Roberts and Dunne (1988), Spanos (1981). Much interest recently has centered on its use in the analysis of hysteretic structures, where differential formulations of hysteresis are combined with the normal equations of motion.

Numerous studies have shown that equivalent linearisation can give good estimates of the mean and mean square of the response. However, since it is inherent in the method to assume that the response is Gaussian, no information on the departures

from Gaussianity, due to non-linear effects, can be gleaned from this approach.

5.3 Equivalent non-linearisation

Instead of replacing the original set of non-linear differential equations with an equivalent *linear* set, one can choose a replacement set of non-linear equations, which belong to a class of problems which can be solved exactly. This approach is referred to "equivalent non-linearisation". The basic idea is due to Caughey (1986).

As an example of this method, consider an oscillator with the following equation of motion:

$$\ddot{x} + b(x,\dot{x}) + g(x) = f(t) \tag{84}$$

where $f(t)$ is a zero-mean, stationary white noise process. One can replace this with

$$\ddot{x} + \dot{x}\, H(E) + g(x) = f(t) \tag{85}$$

An exact solution exists for equation (85), as shown earlier. The error between (84) and (85)

$$\epsilon = b(x,\dot{x}) - \dot{x}\, H(E) \tag{86}$$

may be minimised in a least-square sense, as in normal statistical linearisation.

One needs to choose the function $H(E)$ in an appropriate way. One approach, suggested by Caughey (1986)), is to set

$$H(E) = c\, H_0(E) \tag{87}$$

where $H_0(E)$ is a suitably chosen function of E, and c is a constant, to be determined by the minimisation condition

$$\frac{dE\{\epsilon^2\}}{dc} = 0 \tag{88}$$

It is easy to show from this that

$$c = \frac{E\{\dot{x}\, H_0(E)\, b(x,\dot{x})\}}{E\{\dot{x}^2\, H_o^2(E)\}} \tag{89}$$

It is noted that, with $H_0(E) = 1$, the above reduces to

$$c = \frac{E\{\dot{x}\, b(x,\dot{x})\}}{E\{\dot{x}^2\}} \tag{90}$$

which is the usual statistical linearisation result.

If the damping is of the power-law form

$$b(x,\dot{x}) = b_{r\rho}|x^r\, \dot{x}^s|\mathrm{sgn}(\dot{x}) \tag{91}$$

then a suitable form for $H_0(E)$ is

$$H_0(E) = (2E)^{\frac{r+s-1}{2}} \tag{92}$$

This ensures that $\dot{x}\, H(E)$ is of the same order in x as $b(x,\dot{x})$.

Calculations are much easier in the case of linear stiffness - i.e., non-linearity in damping only. Specific examples are given by Caughey (Caughey (1986)).

An alternative approach is possible which is appropriate when the damping is light - i.e. the energy dissipated per "cycle", due to damping, is, on average, a small fraction of the

average energy in a cycle of oscillation. This implies that $Q(E) << 1$. In this situation one can treat the total energy, E , as approximately constant, over one cycle of oscillation, $T(E)$, where $T(E)$ is given by equation (40). Returning to the expression for the equation error, given by equation (86), then the error integral

$$I = \int_0^{T(E)} \epsilon^2 dt = \int_0^{T(E)} [b(x,\dot{x}) - \dot{x}\, H(E)]^2 dt \tag{93}$$

can be minimised with respect to $H(E)$, where the latter is treated as a constant. This yields

$$H(E) = \frac{\int_0^{T(E)} b(x,\dot{x})\dot{x}dt}{\int_0^{T(E)} \dot{x}^2 dt} \tag{94}$$

or

$$H(E) = \frac{\int_0^{b} b(x,\sqrt{2[E-V(x)]})dx}{\int_0^{b} \sqrt{2[E-V(x)]}dx} \tag{95}$$

where the upper limit, b , is given by $V(b) = E$, as before. A combination of equation (95) with the exact solution of equation (85), as given by equation (83), now gives a solution to the original problem.

One can expect that this solution will become increasingly accurate if the magnitude of the damping is progressively reduced. It will be shown later that this result is in exact agreement with a result obtained by the method of stochastic averaging.

The integrals in equation (95) can be evaluated, either analytically, or numerically, for specific functions, $b(x,\dot{x})$

and $g(x)$, in the equation of motion. The basic result can also be obtained by an energy balance method.

5.4 Closure methods

Suppose that the equations of motion are written in the state variable form given by equation (57). As pointed out earlier, this form is appropriate if the excitation processes can be modelled adequately as white noise processes.

If $g(\underset{\sim}{z})$ is a scalar function of $\underset{\sim}{z}$ one has an important result from Markov process theory (see Jazwinskii (1970)), as follows)

$$E\{\dot{g}(\underset{\sim}{z})\} = E\{\underset{\sim}{h}^T \underset{\sim}{a}\} + \frac{1}{2} \text{tr } E\{\underset{\sim\sim}{D}\underset{\sim\sim}{H}\} \tag{96}$$

where

$$\underset{\sim}{h} = \underset{\sim}{\nabla} g = \left\{\frac{\partial g}{\partial z_1}, \frac{\partial g}{\partial z_2}, \ldots \frac{\partial g}{\partial z_n}\right\}^T \tag{97}$$

and

$$H = \left[\frac{\partial^2 g}{\partial z_i \partial z_j} \right] \tag{98}$$

is the Jacobian matrix of second partial derivatives of g . $\underset{\sim}{D}$ is the diffusion matrix defined by equation (63).

With appropriate choices for the function g , a set of moment equations can be generated from equation (96). It is found that the differential equation for a moment of any order generally involves moments of higher-order, due to the presence of non-linearities. Thus, to obtain a soluble system of equations it is necessary to introduce a "closure scheme". This is the famous "closure problem" in stochastic non-linear mechanics. Two possibilities are

(a) cumulant closure

(b) quasi-moment closure

where cumulants and quasi-moments, of any order, may be related to the ordinary moments

$$m_{k_1 \ldots k_N} = E\left\{ \prod_{i=1}^{N} z_i^{k_i} \right\} \tag{99}$$

where

$$K = k_1 + k_2 + \ldots + k_N \tag{100}$$

is the "order" of the moment (see Stratonovich (1964)).

Closure methods involve setting all cumulants (or quasi-moments), above a specified order, to zero. Considering quasi-moment closure, for example, if all quasi-moments of order greater than M (say) are set to zero, the probability density function for $\underset{\sim}{z}$ may be expressed as (Stratonovich (1964))

$$f_M(\underset{\sim}{z}) = \left\{ 1 - \sum_{s=3}^{M} \frac{1}{s!} \sum_{k_1, k_2 \ldots, k_s = 1}^{N} b_{k_1, k_2 \ldots k_s} H_{k_1, k_2 \ldots k_s}(\underset{\sim}{z} - \underset{\sim}{m}) \right\} f_G(\underset{\sim}{z}) \tag{101}$$

where $f_G(\underset{\sim}{z})$ is the Gaussian density function

$$f_G(\underset{\sim}{z}) = \frac{1}{(2\pi)^{\frac{N}{2}} |\underset{\sim}{V}|} \exp\left\{ -\frac{1}{2} \underset{\sim}{z}_0^T \underset{\sim}{V}^{-1} \underset{\sim}{z}_0 \right\} \tag{102}$$

$$\underset{\sim}{z}_0 = \underset{\sim}{z} - \underset{\sim}{m} \tag{103}$$

$$\underset{\sim}{m} = E\{\underset{\sim}{z}\} \tag{104}$$

and

$$\underset{\sim}{V} = E\{\underset{\sim}{z}_0 \underset{\sim}{z}_0^T\} \tag{105}$$

is the covariance matrix. In equation (101) $b_{k_1 \ldots k_M}$ are the quasi-moments - for low orders ($K \leq 5$) they are identical to cumulants.

Gaussian closure consists of setting all quasi-moments (or cumulants) of order greater than two equal to zero - i.e. it is second order closure. For non-parametric systems Gaussian closure coincides completely with normal statistical linearisation. Thus closure at an order greater than two represents a generalisation of normal statistical linearisation and can be expected to yield an improvement in accuracy. If sufficient moments of the response are generated then it should be possible to use an expansion of $f(\underset{\sim}{z})$, such as that given by equation (101) to estimate the behaviour of the distribution in the tails, with special regard to the influence of non-linearities. Unfortunately, however, the convergence properties of such expansions, at large values of $|\underset{\sim}{z}|$, are such that it is not easy to obtain accurate results by this approach.

5.5 Perturbation methods and functional series

If non-linearities in the system are sufficiently weak then it is possible to generalise the usual perturbation method for non-linear systems to the stochastic case (e.g. see Roberts (1981)), to yield estimates of the influence of the non-linear terms on response statistics. However the work involved in computing the terms in the perturbation expansion can be prohibitive, and it is usual to compute only the first, non-linear term in the expansion.

sions can become prohibitive and results obtained so far by this approach are of rather limited scope.

5.6 Simulation

Finally, mention is made briefly of the method of digital simulation. Here one generates sample functions of the excitation process and corresponding sample functions of the response, by a numerical solution of the equation of motion. Statistical processing of the output process then yields the required information. Lengthy processing is needed to reduce the statistical uncertainty to acceptable limits. The method is simplified for stationary, ergodic processes, because only one input (and output) sample function, of sufficient duration, need be generated. For non-stationary problems an ensemble averaging procedure is necessary.

Simulation methods have the advantage of being very flexible; complex problems, which are impossible to study by analytical methods, can be tackled. However, this approach yields no physical insight into the problem under investigation and in some cases can lead to prohibitive computational requirements.

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AVERAGING METHODS IN RANDOM VIBRATION

J. B. Roberts

CHAPTER TWO

The Stochastic Averaging Method

1. Introduction

It was pointed out in the first chapter that, whilst Markov methods offer a powerful general methodology for tackling random vibration problems of the non-linear kind, unfortunately it is possible to find exact solutions in only a few special cases. Numerical solutions are, of course, possible but these tend to involve considerable computing effort.

In this Chapter a powerful approximate technique is introduced, which is particularly suitable when the damping in the system is light. This condition frequently occurs in practical non-linear vibration problems. The method was originally proposed by Stratonovich (1964) as a mean of obtaining results for non-linear, self-excited oscillations in the presence of noise. It has subsequently been examined mathematically by various workers (Khasminskii (1966,1968), Papanicolaou (1973), Papanicolaou and Kohler (1974)), with a view to establishing a rigorous basis. The essence of the method is embodied in the so-called "Stratonovich-Khasminskii limit theorem", which will be introduced later in this Chapter. The technique may be viewed as an extension to the stochastic case of the well-known Bogoliubov and Mitropolsky method for approximately solving deterministic non-linear vibration problems (see Bogoliubov and Mitropolsky (1961)).

As a means of providing a simple introduction to the method, with a minimum of mathematical complexity, we will first consider the case of an oscillator with non-linear damping and linear stiffness. It will be shown that, if the damping is sufficiently light, and the excitation is broad-band, then the amplitude process, $a(t)$, of such an oscillator can be modelled as a one-dimensional Markov process. The solution of the FPK equation yields the distribution of $a(t)$, and, more importantly, the joint distribution of displacement and velocity of the response. From this various statistics of the response, relevant to reliability, such as level crossing rates, can be computed fairly easily.

Following this discussion it will be shown that the method can be extended fairly easily to oscillators with parametric excitation, and to multi-degree of freedom systems.

It is noted that the application of stochastic averaging to mechanical and structural random vibration problems has been reviewed recently by Roberts and Spanos (1986).

2. Oscillators with non-linear damping

The simplest systems of concern in engineering are oscillators with a single degree of freedom. Consider the following equation of motion of a randomly excited non-linear oscillator:

$$\ddot{x} + \epsilon^2 h(x,\dot{x}) + \omega_0^2 x = f(t) \qquad (1)$$

For $\epsilon^2 = 0$ this equation reduces to that of an undamped oscillator, with a natural frequency of oscillation, ω_0 . The scaling parameter, ϵ^2 , is introduced here as a means of quantifying the strength of the non-linear term, $h(x,\dot{x})$.

It will be assumed that $f(t)$ is a stationary, broad-band process, with zero mean, which possesses a power spectrum, $S_f(\omega)$. Here $S_f(\omega)$ is defined by

$$S_f(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w_f(\tau)\cos\omega\tau d\tau \tag{2}$$

where $w_f(\tau)$ is the correlation function for $f(t)$, defined by

$$w_f(\tau) = E\{f(t)f(t+\tau)\} \tag{3}$$

Attention will now be restricted to the situation where the damping is *light* - i.e., ϵ^2 is *small*. Moreover, it will be assumed that the standard deviation of the stationary response, σ , is such that

$$\sigma = 0(\epsilon^{-1}) \tag{4}$$

This is certainly true in the linear case, as can easily be shown by standard linear theory. Thus, if

$$h(x,\dot{x}) = \dot{x} \tag{5}$$

then

$$\sigma^2 = \int_{-\infty}^{\infty} \frac{S_f(\omega)\,d\omega}{|(\omega_0^2-\omega^2) + i\epsilon^2\omega|^2} \tag{6}$$

and as $\epsilon^2 \to 0$

$$\sigma^2 \to \frac{\pi S_f(\omega_0)}{\epsilon^2\omega_0^2} \tag{7}$$

Equation (4) also appears to be generally true in the non-linear case, although a rigorous proof seems to be difficult. It is noted, however, that the form of the damping function will not

influence the order of the response, with respect to ϵ , so one can expect equation (4) to be generally valid.

It follows from equation (6) that

$$\sigma^2 \rightarrow \infty \qquad \text{as} \qquad \epsilon \rightarrow 0 \tag{8}$$

For analysis purposes this is inconvenient, since the limiting behaviour of the response, as the magnitude of the damping tends to zero, will be of concern here. It is desirable, therefore, to *scale* the excitation so that the level of the response, as measured by σ , is of order ϵ^0 . This implies that the excitation spectrum should be of order ϵ^2 - i.e.,

$$\underset{\text{over } \omega}{\text{Sup}}\{S_f(\omega)\} = O(\epsilon^2) \tag{9}$$

This scaling can be made explicit by introducing the process $z(t)$, where

$$f(t) = \epsilon z(t) \tag{10}$$

The equation of motion now becomes

$$\ddot{x} + \epsilon^2 h(x,\dot{x}) + \omega_0^2 x = \epsilon z(t) \tag{11}$$

It is important to appreciate that this step does not imply any restrictions on the "strength" of the excitation process. When ϵ^2 is small the level of the excitation will normally be weak compared with the maximum level of the response: in other words, for light damping, the response will grow until its level is large, in some sense, compared with the level of the excitation. The scaling in equation (10) simply makes this feature appear explicitly.

2.1 Transformation of variables

Consider the total energy of the oscillator, $E(t)$, defined by

$$E(t) = \frac{\dot{x}^2}{2} + \frac{\omega_0^2 x^2}{2} \tag{12}$$

where the first term on the left-hand side of this equation is the kinetic energy, and the second term is the potential energy (see also section 4 of Chapter 1). If equation (11) is multiplied throughout by $\dot{x}$ one has

$$\dot{x}\ddot{x} + \epsilon^2 \dot{x} h(x,\dot{x}) + \omega_0^2 \dot{x} x = \epsilon \dot{x} z(t) \tag{13}$$

Moreover, differentiating equation (12) throughout with respect to time gives

$$\dot{E} = \dot{x}\ddot{x} + \omega_0^2 \dot{x} x \tag{14}$$

On combining equations (13) and (14) one has

$$\dot{E} = P_{in}(t) - P_{dis}(t) \tag{15}$$

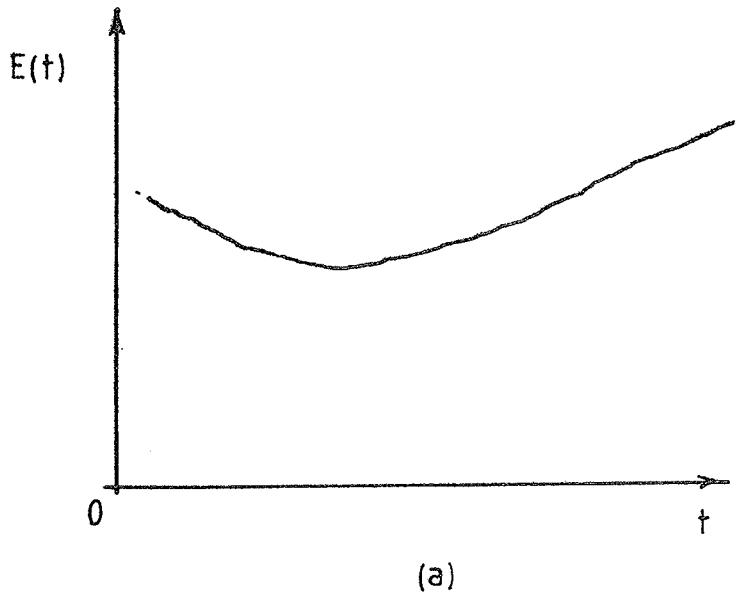
where

$$P_{in}(t) = \epsilon \dot{x} z(t) \tag{16}$$

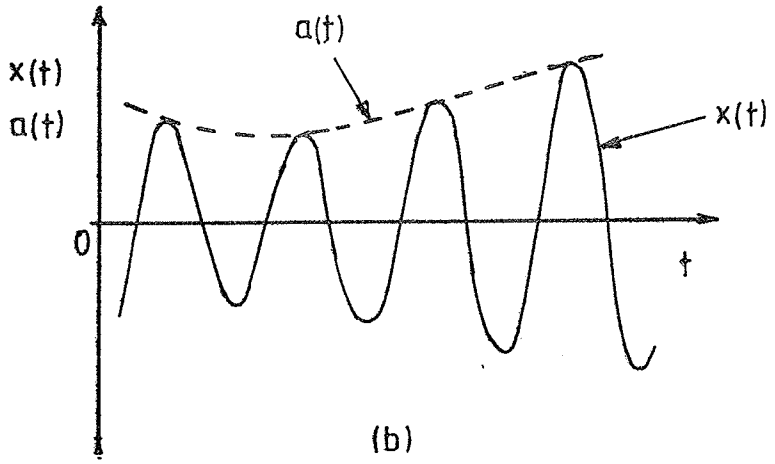
and

$$P_{dis}(t) = \epsilon^2 \dot{x} h(x,\dot{x}) \tag{17}$$

Equation (15) is simply a power balance equation. In words, it states that the rate of change of the total energy of the oscillator, with respect to time, is equal to the power input due to the random excitation, $P_{in}(t)$, minus the power dissipation due to the damping mechanism, $P_{dis}(t)$. Since ϵ is here taken to be small, it follows from equation (15) that $\dot{E}$ will also be small - i.e. sample functions of $E(t)$ will be varying slowly



(a)



(b)

Fig. 2.1.

with time, as illustrated in Fig. 2.1(a). This may be compared with a sample function of x(t) , which will be oscillatory, due to the narrow-band character of this process, see Fig. 2.1(b). (Strictly, E(t) sample functions may be only slowly varying in a *macroscopic* sense; at a more detailed level, microscopic irregularities may appear. Fortunately this feature need not concern us here).

The principal objective of the stochastic averaging method is to average both $P_{in}(t)$ and $P_{dis}(t)$ over time, using the fact that, for small ϵ, $E(t)$ is approximately constant over a period of time corresponding to the natural period of oscillation

$$T = 2\pi/\omega_0 \tag{18}$$

We shall see that such an averaging procedure leads to a one-dimensional Markov model for $E(t)$, and hence to an FPK equation for the transition density function of this process.

In the present example it is convenient to work with an *amplitude process* $a(t)$, rather than $E(t)$, where $a(t)$ is defined by

$$E(t) = V[a(t)] \tag{19}$$

and $V(\cdot)$ is the potential energy function of the oscillator here given by

$$V(x) = \frac{\omega_0^2 x^2}{2} \tag{20}$$

Hence

$$E(t) = \frac{\omega_0^2 a^2}{2} \tag{21}$$

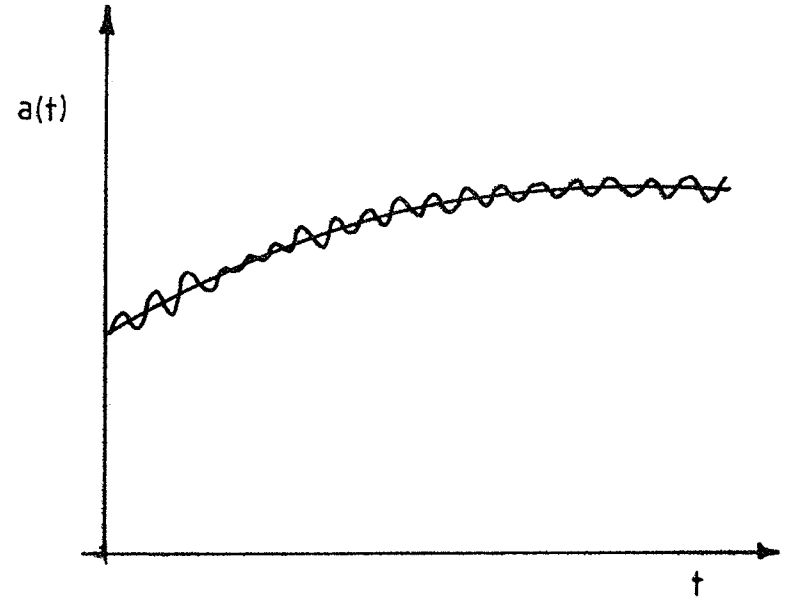
$a(t)$ has the same dimensions as $x(t)$ and, when ϵ is small, will "follow" the peaks of the response process, $x(t)$. Clearly, from equation (19), if $E(t)$ is slowly varying then so is $a(t)$: hence the latter process can be averaged using the concept previously described in connection with $E(t)$.

Associated with $a(t)$ is a phase process, $\phi(t)$; both $a(t)$ and $\phi(t)$ may be related to $x(t)$ through the following "van der Pol transformation":

$$x(t) = a(t)\cos(\omega_0 t + \phi) \tag{22}$$

$$\dot{x}(t) = -a(t)\omega_0 \sin(\omega_o t + \phi) \tag{23}$$

It is easy to see, from this definition of $a(t)$ and $\phi(t)$, that $a(t)$ is consistent with equation (21).



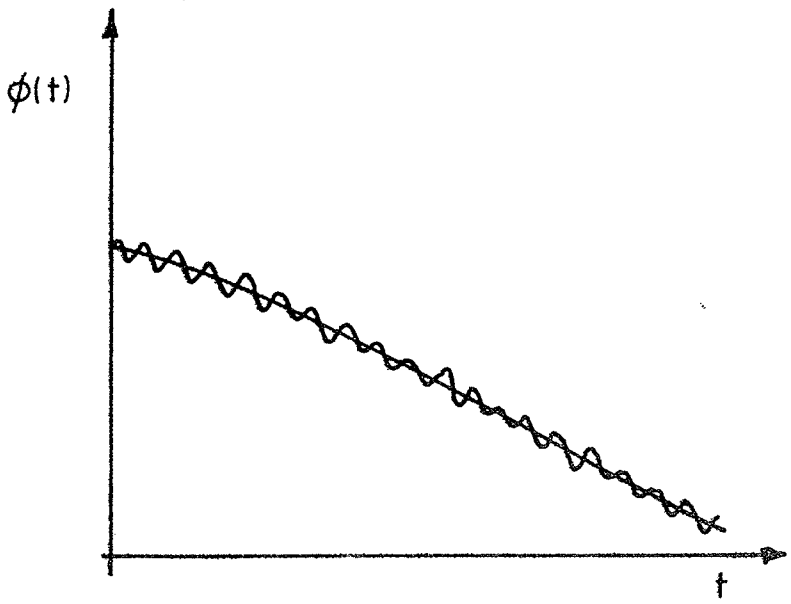


Fig. 2.2.

When ϵ is small, *both* a(t) and $\phi(t)$ are slowly varying. This can be seen by rewriting the original equation of motion in terms of a and ϕ (see equation (11)). This gives

$$\dot{a} = \frac{\epsilon^2}{\omega_0} h(a\cos\Phi, -a\omega_0\sin\Phi)\sin\Phi - \frac{\epsilon z(t)}{\omega_0}\sin\Phi \tag{24}$$

$$\dot{\phi} = \frac{\epsilon^2}{a\omega_0} h(a\cos\Phi, -a\omega_0\sin\Phi)\cos\Phi - \frac{\epsilon z(t)}{a\omega_0}\cos\Phi \tag{25}$$

where

$$\Phi = \omega_0 t + \phi \tag{26}$$

The right-hand sides of equations (24) and (25) depend not only on a(t), $\phi(t)$ and z(t) but also explicitly on time, through the "oscillatory" terms $\sin\Phi$ and $\cos\Phi$. These terms produce small, relatively rapid fluctuations superimposed on relatively large, but slowly varying fluctuations in a(t) and $\phi(t)$, as illustrated in Fig. 2.2.

The basic idea of the stochastic averaging method is to eliminate the fluctuational terms by performing suitable time averaging. In this way one can simplify equations (24) and (25) very considerably, as we shall now demonstrate.

2.2 Averaging the dissipation term

Consider the special case where the excitation is absent (z(t) = 0) . Equations (24) and (25) reduce to

$$\dot{a} = \frac{\epsilon^2}{\omega_0} h(a\cos\Phi, -a\omega_0\sin\Phi)\sin\Phi \tag{27}$$

$$\dot{\phi} = \frac{\epsilon^2}{a\omega_0} h(a\cos\Phi, -a\omega_0\sin\Phi)\cos\Phi \tag{28}$$

The right hand sides of these equations can be averaged by assuming that a and ϕ remain approximately constant, over one cycle, of period T, as given by equation (18). If

$$F(a) = -\frac{1}{2\pi}\int_0^{2\pi} h(a\cos\Phi, -a\omega_0\sin\Phi)\sin\Phi \, d\Phi \tag{29}$$

and

$$G(a) = -\frac{1}{2\pi}\int_0^{2\pi} h(a\cos\Phi, -a\omega_0\sin\Phi)\cos\Phi \, d\Phi \tag{30}$$

then the averaged equations, corresponding to (27) and (28) may be written simply as

$$\dot{a} = -\frac{\epsilon^2}{\omega_0} F(a) \tag{31}$$

$$\dot{\phi} = -\frac{\epsilon^2}{a\omega_0} G(a) \tag{32}$$

It can be seen that, in the simplified equations, the equation for $a(t)$ is *uncoupled* from that of $\phi(t)$. Thus the first order equation for $a(t)$ can be solved *independently* : this, as will be seen, is a basic feature of the averaging method and still applies when the excitation is present. In many cases the damping is an odd function of velocity, such that $G(a)$ is zero; then $\dot{\phi} = 0$, implying that ϕ is a constant.

As a specific example, consider the case of an oscillator with linear damping. Then

$$\epsilon^2 h(x,\dot{x}) = 2\zeta\omega_0\dot{x} \tag{33}$$

where ζ is the usual non-dimensional damping factor, and it follows from equation (29) that

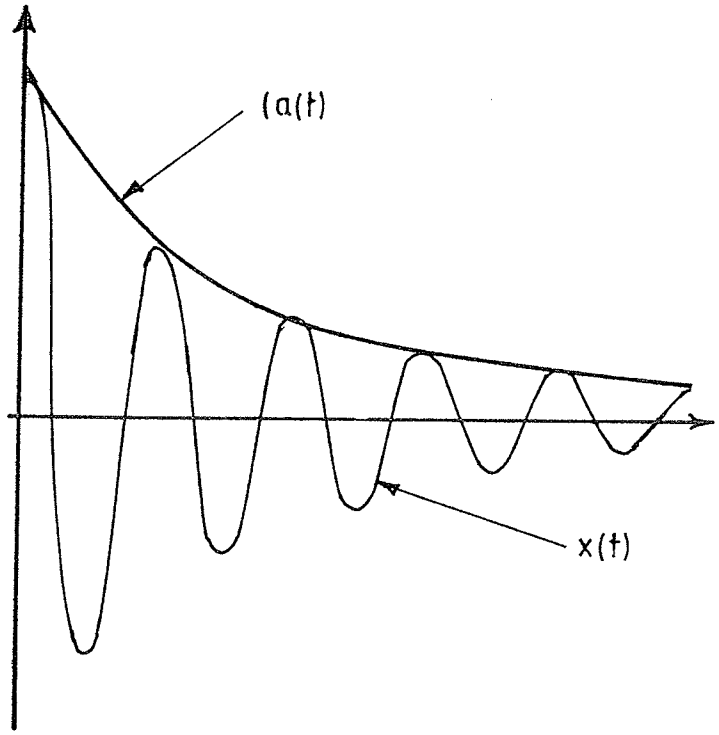


Fig. 2.3.

$$F(a) = -\frac{2\zeta\omega_0}{2\pi}\int_0^{2\pi}(-a\omega_0\sin\Phi)\sin\Phi\, d\Phi \tag{34}$$

Hence

$$F(a) = \zeta\omega_0^2 a \tag{35}$$

Combining equations (31) and (35), and solving one obtains

$$a(t) = a(0)\exp(-\zeta\omega_0 t) \tag{36}$$

This solution is sketched in Fig. 2.3. It can easily be demonstrated, through the exact transient solution for a linear oscillator, sketched in Fig. 2.3, that $a(t)$ is an extremely good representation of the envelope of the free-decay response to some initial disturbance.

It is noted that, since $\dot{a}$ is proportional to ϵ^2, the error involved in treating a and ϕ as constants, over a

period T, is of order ϵ^2. Thus equations (31) and (32) are correct to order ϵ^2, with an error of order ϵ^4.

Returning to the more general case where excitation is present (see equations (24) and (25)) it is noted that the first terms on the right hand side may be averaged in the manner indicated above, whether or not random excitation is present. Thus partially simplified forms for equations (24) and (25) are as follows:

$$\dot{a} = -\frac{\epsilon^2}{\omega_0} F(a) - \frac{\epsilon z(t)}{\omega_0} \sin\Phi \tag{37}$$

$$\dot{\phi} = -\frac{\epsilon^2}{\omega_0} G(a) - \frac{\epsilon z(t)}{a\omega_0} \cos\Phi \tag{38}$$

2.3 Averaging the excitation term

So far the averaging method corresponds exactly with the deterministic method of Bogoliubov and Mitropolsky. The next step, however, which involves applying averaging to the last terms in equations (37) and (38), is rather more difficult. The correct procedure for averaging those excitation terms is due to Stratonovich (1964). Here we give a simplified approach to the averaging method which, although not rigorous, does give a physical insight into the nature of the approximations involved.

The basic difficulty in averaging the terms

$$y_1(t) = \frac{1}{\omega_0} z(t) \sin\Phi \tag{39}$$

$$y_2(t) = \frac{z(t)}{a(t)\omega_0} \cos\Phi \tag{40}$$

which appear in equations (37) and (38) lies in the fact that the phase process, $\phi(t)$, is correlated with $z(t)$. Thus the means

of $y_1(t)$ and $y_2(t)$ are generally non-zero, even though the processes $z(t)$ and $\phi(t)$ have zero mean.

Suppose that the excitation process $z(t)$ has a correlation time scale, τ_{cor}, which is so small that one can define a time interval Δt, such that the following two conditions are satisfied, simultaneously:

(i) $\Delta t >> \tau_{cor}$

(ii) $a(t)$ and $\phi(t)$ do not change appreciably from t to $t + \Delta t$.

It is noted that these conditions are always met if ϵ is sufficiently small, for then the correlation time scale associated with the response will always become large, compared with that of the excitation. Put another way, the bandwidth of the response will become small, compared with that of the excitation. From a practical viewpoint, however, it is desirable that the excitation should have a large band-width (so that τ_{cor} is, alsolutely, small), otherwise ϵ may have to be extremely small (and maybe unrealistically so) in order that the above requirements be met.

If the above conditions hold then, with an error of order $(\Delta\phi)^2$, one can write

$$\sin(\omega_0 t+\phi) = \sin(\omega_o t+\phi_1) + \cos(\omega_0 t+\phi_1)\Delta\phi \qquad (41)$$

where

$$\left.\begin{aligned} \phi_1 &= \phi(t-\Delta t) \\ \phi &= \phi(t) \\ \Delta\phi &= \phi-\phi_1 \end{aligned}\right\} \qquad (42)$$

If this expression is substituted into equation (39) one obtains

$$y_1(t) = \frac{1}{\omega_0} z(t) \left[\sin(\omega_0 t+\phi_1) + \cos(\omega_0 t+\phi_1)\Delta\phi\right] \quad (43)$$

Now, since ϕ_1 is uncorrelated with $z(t)$, $\Delta t >> \tau_{cor}$, the average value of $z(t)\sin(\omega_0 t+\phi_1)$ is zero. Hence the mean of $y_1(t)$ is given by

$$E\{y_1(t)\} = \frac{1}{\omega_0} \cos(\omega_0 t+\phi_1)\ E\{z(t)\Delta\phi\} \quad (44)$$

An expression for $\Delta\phi$ can be found by integrating the phase equation, as given by equation (38). Thus

$$\Delta\phi = -\frac{\epsilon^2}{\omega_0} G(a)\Delta t - \frac{\epsilon}{a\omega_0} \int_{t-\Delta t}^{t} z(\tau)\ \cos(\omega_0\tau+\phi)\,d\tau \quad (45)$$

where we have used the fact that $a(t)$ and $\phi(t)$ are sensibly constant over the interval Δt. A combination of equations (44) and (45) now gives

$$E\{y_1(t)\} = \frac{-\epsilon}{a\omega_0^2} \int_{t-\Delta t}^{t} E\{z(t)z(\tau)\}\cos(\omega_0 t+\phi)\cos(\omega_0\tau+\phi)\,d\tau \quad (46)$$

With a change of variables

$$u = \tau - t \quad (47)$$

equation (46) can be re-expressed as

$$E\{y_1(t)\} = \frac{-\epsilon}{a\omega_0^2} \int_{-\infty}^{0} w_z(u)\ \cos(\omega_0 t+\phi)\cos(\omega_0 t+\omega_0 u+\phi)\,du \quad (48)$$

where

$$w_z(\tau) = E\{z(t)z(t+\tau)\} \quad (49)$$

is the correlation function for $z(t)$, and the bottom limit of the integral, Δt , has been replaced by $-\infty$, since the correlation function is effectively zero when $u = \Delta t$ ($\tau_{cor} << \Delta t$, again).

The product of the two cosines in equation (48) can be written as

$$\frac{1}{2}\cos(\omega_0 u) + \text{oscillatory terms, with frequency } \omega_0 \qquad (50)$$

Substituting this into equation (48) gives

$$E\{y_1(t)\} = \frac{-\epsilon}{2a\omega_0^2}\int_{-\infty}^{0} w_z(u)\cos(\omega_0 u)du + \text{oscillatory terms} \qquad (51)$$

The oscillatory terms, will, when averaged over the period, T , be zero. Moreover, the power spectrum, $S_z(\omega)$, of $z(t)$, is defined by

$$S_z(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w_z(u)\cos(\omega u)du \qquad (52)$$

It follows that

$$\bar{y}_1 = E\{y_1(t)\} = \frac{-\epsilon\pi S_z(\omega_0)}{2a\omega_0^2} \qquad (53)$$

Next, the correlation function of the zero mean process

$$y_1^* = y_1 - \bar{y}_1 \qquad (54)$$

will be found. Since the second term on the right-hand side of equation (43) influences only the mean value, $\bar{y}_1$, one need only consider the contribution from the first term, in evaluating the correlation function of y_1 . Thus

$$y_1^* = z(t)\ \sin(\omega_0 t + \phi_1) \tag{55}$$

and

$$w_{y_1}(\tau) = E\{y_1^*(t)\ y_1^*(t+\tau)\}$$

$$= E\{z(t)z(t+\tau)\sin(\omega_0 t+\phi)\sin(\omega_0 t+\omega_0\tau+\phi)\} \tag{56}$$

where the fact that ϕ is sensibly constant, over an interval where the correlation function of $z(t)$ is significant, has been used.

Now since τ_{cor} is much smaller than the correlation time scale of the response, the process y_1^* behaves approximately like an uncorrelated white noise process. Thus one can approximate $w_{y_1}(\tau)$ as

$$w_{y_1}(\tau) = I\ \delta(\tau) \tag{57}$$

where the "strength" of the white noise process is given by

$$I = \int_{-\infty}^{\infty} w_{y_1}(\tau)\,d\tau \tag{58}$$

Combining equations (56) and (58) one has

$$I = \int_{-\infty}^{\infty} w_z(\tau)\sin(\omega_0 t+\phi)\sin(\omega_0 t+\omega_0\tau+\phi)\,d\tau \tag{59}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} w_z(\tau)\cos(\omega_0\tau)\,d\tau + \text{oscillatory terms} \tag{60}$$

Once again, the oscillatory terms will disappear if averaging is carried out, over the period T. Also, the interval in the above expression can be related to the power spectrum, $S_z(\omega)$, of $z(t)$. Thus, from equations (52) and (60)

$$I = \pi S_z(\omega_0) \tag{61}$$

The above analysis has shown that the process $y_1(t)$ can be approximated as a white noise process, with a strength I given by equation (61) and a non-zero mean, given by equation (53). Hence one can express $y_1(t)$ as

$$y_1(t) = -\frac{\epsilon\pi S_z(\omega_0)}{2a\omega_0} - [\pi S_z(\omega_0)]^{\frac{1}{2}}\, \xi_1(t) \tag{62}$$

where $\xi_1(t)$ is a zero-mean white noise process, with unit strength. Combining equations (37), (39) and (62) one obtains

$$\dot{a} = -\frac{\epsilon^2}{\omega_0} F(a) + \frac{\epsilon^2\pi S_z(\omega_0)}{2a\omega_0^2} + \frac{[\epsilon^2\pi S_z(\omega_0)]^{\frac{1}{2}}}{\omega_0}\, \xi_1(t) \tag{63}$$

This is the final, "simplified equation" for $a(t)$.

A similar analysis can be carried out for the phase process, $\phi(t)$, to show that

$$\dot{\phi} = -\frac{\epsilon^2}{a\omega_0} G(a) + \frac{[\epsilon^2\pi S_z(\omega_0)]^{\frac{1}{2}}}{a\omega_0}\, \xi_2(t) \tag{64}$$

where ξ_2 is another white noise process, with zero mean and unit strength, independent of ξ_1 . It is observed that the mean of the process $y_2(t)$, defined by equation (40) is zero. Equation (64) is the "simplified equation" for $\phi(t)$.

From equation (10) it follows that the spectrum of $f(t)$ is given by

$$S_f(\omega) = \epsilon^2 S_z(\omega) \tag{65}$$

Thus, in terms of $S_f(\omega)$, one can write the simplified equations as

$$\dot{a} = -\frac{\epsilon^2}{\omega_0} F(a) + \frac{\pi S_f(\omega_0)}{2a\omega_0^2} + \frac{[\pi S_f(\omega_0)]^{\frac{1}{2}}}{\omega_0} \xi_1(t) \tag{66}$$

$$\dot{\phi} = -\frac{\epsilon^2}{a\omega_0} G(a) + \frac{[\pi S_f(\omega_0)]^{\frac{1}{2}}}{\omega_0} \xi_2(t) \tag{67}$$

Written rather more formally, as Itô equations, equations (63) and (64) become (see Chapter 1)

$$d\underset{\sim}{X} = \epsilon^2 \underset{\sim}{a}(\underset{\sim}{x})dt + \epsilon \underset{\sim}{B}\, d\underset{\sim}{W} \tag{68}$$

where

$$\underset{\sim}{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} a \\ \phi \end{bmatrix} \tag{69}$$

$$\underset{\sim}{a} = \begin{bmatrix} a_1(\underset{\sim}{X}) \\ a_2(\underset{\sim}{X}) \end{bmatrix} \quad , \quad \underset{\sim}{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{70}$$

$$d\underset{\sim}{W} = \begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix} \tag{71}$$

and

$$a_1(\underset{\sim}{X}) = -\frac{F(a)}{\omega_0} + \frac{\pi S_z(\omega_0)}{2a\omega_0^2} \tag{72}$$

$$a_2(\underset{\sim}{X}) = -\frac{G(a)}{a\omega_0^2} \tag{73}$$

$$B_{11} = \frac{[\pi S_z(\omega_0)]^{\frac{1}{2}}}{\omega_0} = a\, B_{22} \qquad (74)$$

and $B_{12} = B_{21} = 0$. W_1 and W_2 are independent Wiener processes, and formally

$$\frac{dW_1}{dt} = \xi_1 \quad , \quad \frac{dW_2}{dt} = \xi_2 \qquad (75)$$

It can be shown (see Stratonovich (1964)) that the approximations involved in averaging the stochastic terms, as outlined above, are entirely consistent with the approximations inherent in averaging the dissipation term. In fact the averaging approximation is, overall, correct to order ϵ^2 .

Since $\underset{\sim}{X}$, as defined by equation (69) is governed by a first-order equation, with white noise excitation, it follows (see Chapter 1) that the joint process $[a(t),\phi(t)]$ is (at least approximately) a joint Markov process. Strictly, $[a(t),\phi(t)]$ converges to a Markov process as ϵ^2 tends to zero. Thus one can regard the simplified equations for $a(t)$ and $\phi(t)$ as asymptotically exact, as $\epsilon \to 0$. The mathematical basis for this result is contained in the Stratonovich-Khasminskii limit theorem.

2.4 The Stratonovich-Khasminskii limit theorem

This theorem is applicable to stochastic differential equations of the following form

$$\dot{\underset{\sim}{X}} = \epsilon^2 \underset{\sim}{f}(\underset{\sim}{X},t) + \epsilon \underset{\sim}{g}[\underset{\sim}{X},t,\underset{\sim}{Y}(t)] \qquad (76)$$

Here $\underset{\sim}{X}(t)$ is an n-vector stochastic process, usually relating to the response, and $\underset{\sim}{Y}(t)$ is an m-vector stochastic process, usually relating to the excitation. The elements of $\underset{\sim}{Y}(t)$ are broad-band processes, with zero-means, and the vectors $\underset{\sim}{f}$ and $\underset{\sim}{g}$ satisfy certain requirements (Khasminskii (1966)) which are

almost invariably met in practice. It can be shown that $\underset{\sim}{X}(t)$ may be uniformly approximated over a time interval of order ϵ^{-1} by an n-dimensional Markov process, which satisfies the Itô equation

$$d\underset{\sim}{X} = \epsilon^2 \underset{\sim}{a}(\underset{\sim}{X})dt + \epsilon \underset{\sim}{B}(\underset{\sim}{X})d\underset{\sim}{W} \tag{77}$$

Here the symbols $\underset{\sim}{W}$ denotes an n-vector of independent unit Wiener processes and $\underset{\sim}{a}$ and $\underset{\sim}{B}$ are, respectively, the "drift vector", and "diffusion matrix". In fact, according to the limit theorem, $\underset{\sim}{X}(t)$ converges weakly to a Markov process as $\epsilon \to 0$. Hence the approximation of $\underset{\sim}{X}(t)$ as a Markov process, governed by equation (77), is asymptotically exact, as $\epsilon \to 0$.

The quantities $\underset{\sim}{a}$ and $\underset{\sim}{B}$ can be evaluated according to the following expressions:

$$\underset{\sim}{a} = T^{av}\left[E\{\underset{\sim}{f}\} + \int_{-\infty}^{0} E\left\{\left[\frac{\partial \underset{\sim}{g}}{\partial \underset{\sim}{X}}\right]_t (\underset{\sim}{g})_{t+\tau}\right\}d\tau\right] \tag{78}$$

and

$$\underset{\sim}{B}\,\underset{\sim}{B}^T = T^{av}\int_{-\infty}^{\infty} E\{\underset{\sim}{g}_t\,\underset{\sim}{g}^T_{t+\tau}\}d\tau \tag{79}$$

Here T^{av} is a time-averaging operator, defined by

$$T^{av}\{\cdot\} = \lim_{T\to\infty}\frac{1}{T}\int_{t_0}^{t_0+T}\{\cdot\}dt \tag{80}$$

the integration being performed over explicit t only. Also the subscripts t and $t+\tau$ in the above indicate that the appropriate quantities are to be evaluated at those times. In evaluating the expectations in equations (78) and (79) the elements of $\underset{\sim}{X}(t)$ are treated as constants.

It is noted that, if the expected quantities in equations (78) and (79) are periodic, with period T_0, say, then T^{av} is just on averaging operation over one period; i.e.,

$$T_0^{av}\{\cdot\} = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} \{\cdot\}dt \tag{81}$$

and the result is independent of t_0.

2.5 Application of the limit theorem

The limit theorem, as stated above, may be used as an alternative means of deriving the simplified equations for $a(t)$ and $\phi(t)$.

Returning to the original, exact equations for a and ϕ, as given by equations (24) and (25), it is seen that they can be written in the form of equation (76), if $\underset{\sim}{X}$ is defined in equation (69). Here

$$\underset{\sim}{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad , \quad \underset{\sim}{g} = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \tag{82}$$

where

$$f_1 = \frac{1}{\omega_0} h(a\cos\Phi,\ -a\omega_0\sin\Phi)\sin\Phi \tag{83}$$

$$f_2 = \frac{1}{a\omega_0} h(a\cos\Phi,\ -a\omega_0\sin\Phi)\cos\Phi \tag{84}$$

$$g_1 = \frac{1}{\omega_0} z(t)\sin\Phi \tag{85}$$

$$g_2 = \frac{1}{a\omega_0} z(t)\cos\Phi \tag{86}$$

It follows, immediately, from the limit theorem that, asymptotically, as $\epsilon \to 0$, $\underset{\sim}{X}(t)$ is governed by the Itô equation, as given by equation (68).

The elements of $\underset{\sim}{a}$ and $\underset{\sim}{B}$ may be found through an application of equations (78) and (79). Thus, from equation (78),

$$a_1 = T^{av} E\{f_1\}$$

$$+ T^{av} \int_{-\infty}^{0} E\left\{ \left[\frac{\partial g_1}{\partial X_1}\right]_t (g_1)_{t+\tau} + \left[\frac{\partial g_1}{\partial X_2}\right]_t (g_2)_{t+\tau} \right\} d\tau \qquad (87)$$

In the first term on the left-hand side of the above equation the excitation, $z(t)$, is not present explicitly, so that this term reduces to

$$T_0^{av}\{f_1\} = -\frac{1}{\omega_0} F(a) \qquad (88)$$

where $F(a)$ is given by equation (29), and use has been made of the fact that cyclic averaging, as indicated by equation (81), is required here. From the fore-going definitions of g_1 and g_2 one has

$$\frac{\partial g_1}{\partial X_1} = \frac{\partial g_1}{\partial a} = 0 \qquad (89)$$

$$\frac{\partial g_1}{\partial X_2} = \frac{\partial g_1}{\partial \phi} = \frac{1}{\omega_0} z(t)\cos\Phi \qquad (90)$$

Hence the second part of the right-hand side of equation (87) may be written as

$$\frac{1}{a\omega_0^2} T_{av}\left[\int_{-\infty}^{0} E\left\{ z(t)\cos\Phi_t \; z(t+\tau)\cos\Phi_{t+\tau} \right\} d\tau \right]$$

$$= \frac{1}{a\omega_0^2} T_{av}\left[\int_{-\infty}^{0} w_z(\tau)\cos\Phi_t \cos\Phi_{t+\tau} d\tau \right] \qquad (91)$$

It can be seen that, apart from a factor ϵ, this expression is identical to that found earlier, for $E\{y_1(t)\}$ (see equation (48)); thus, averaging as before to remove the oscillatory terms, one has, from equations (87), (88) and (91),

$$a_1 = -\frac{1}{\omega_0} F(a) + \frac{\pi S_y(\omega_0)}{2a\omega_0^2} \qquad (92)$$

in agreement with equation (72).

The term a_2 is also easily evaluated from equation (78). Thus

$$a_2 = T_0^{av} E\{f_2\}$$

$$+ T_0^{av} \int_{-\infty}^{0} E\left\{ \left[\frac{\partial g_2}{\partial x_1}\right]_t (g_1)_{t+\tau} + \left[\frac{\partial g_2}{\partial x_2}\right]_t (g_2)_{t+\tau} \right\} d\tau \qquad (93)$$

Now

$$T_0^{av} E\{f_2\} = -\frac{1}{a\omega_0} G(a) \qquad (94)$$

where $G(a)$ is defined by equation (30). Moreover,

$$\frac{\partial g_2}{\partial x_1} = \frac{\partial g_2}{\partial a} = -\frac{1}{a^2\omega_0} z(t)\cos\Phi \qquad (95)$$

and

$$\frac{\partial g_2}{\partial x_2} = \frac{\partial g_2}{\partial a} = -\frac{1}{a\omega_0} z(t)\sin\Phi \qquad (96)$$

Hence the second part of the right-hand side of equation (93) becomes

$$- \frac{1}{a^2\omega_0^2} T_0^{av} \left[\int_{-\infty}^{0} w_z(\tau) \sin(\Phi_t + \Phi_{t+\tau}) d\tau \right]$$

$$= - \frac{1}{a^2\omega_0^2} T_0^{av} \left[\sin 2\Phi_t \int_{-\infty}^{0} w_z(\tau) \sin(\omega_0\tau) d\tau \right.$$

$$\left. + \cos 2\Phi_t \int_{-\infty}^{0} w_z(\tau) \cos(\omega_0\tau) d\tau \right] \qquad (97)$$

Now both terms in the square brackets will vanish, when the averaging operation is carried out. Thus one finds that a_2 is simply given by

$$a_2 = - \frac{1}{a\omega_0} G(a) \qquad (98)$$

in agreement with equation (73).

Finally, the elements of $\underset{\sim}{B}$ can be evaluated by the use of equation (79). Here this gives the following three equations for the elements of $\underset{\sim}{B}$:

$$B_{11}^2 + B_{12}^2 = \frac{T_0^{av}}{\omega_0^2} \int_{-\infty}^{\infty} w_z(\tau) \sin\Phi_t \sin\Phi_{t+\tau} d\tau \qquad (99)$$

$$B_{11}B_{21} + B_{12}B_{22} = \frac{T_0^{av}}{a\omega_0^2} \int_{-\infty}^{\infty} w_z(\tau) \sin\Phi_t \cos\Phi_{t+\tau} d\tau \qquad (100)$$

and

$$B_{21}^2 + B_{22}^2 = \frac{T_0^{av}}{a^2\omega_0^2} \int_{-\infty}^{\infty} w_z(\tau) \cos\Phi_t \cos\Phi_{t+\tau} d\tau \qquad (101)$$

Solving one obtains $B_{12} = B_{21} = 0$ and B_{11} and B_{22} in agreement with equation (74).

2.6 The FPK equations

It has been shown that the joint process $[a(t),\phi(t)]$ converges to a Markov process, with the variables governed by Itô equations, in the form of equation (68). It follows from the results given in Chapter One that $[a(t),\phi(t)]$ has a transition density $p(a,\phi|a_0,\phi_0;t)$, such that $p(a,\phi|a_0,\phi_0;t)dad\phi$ is the probability that $a < a(t) < a+da$, $\phi < \phi(t) < \phi+d\phi$, at time t, given by $a = a_0$, $\phi = \phi_0$ at $t = 0$. The transition density function is governed by the following FPK equation (see equations (1.61) and (1.62))

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial a}\left[\left\{ \frac{\epsilon^2 F(a)}{\omega_0} - \frac{\pi S_f(\omega_0)}{2a\omega_0^2} \right\}p\right] + \frac{\epsilon^2 G(a)}{a\omega_0}\frac{\partial p}{\partial a}$$

$$+ \frac{\pi S_f(\omega_0)}{2a\omega_0^2}\left[\frac{\partial^2 p}{\partial a^2} + \frac{1}{a^2}\frac{\partial^2 p}{\partial \phi^2}\right] \qquad (102)$$

An inspection of the differential equations for $a(t)$ and $\phi(t)$ (see equations (66) and (67) shows that the amplitude process, $a(t)$, is *uncoupled* from the phase process, ϕ (as in the deterministic case, where the excitation is absent). It follows that $a(t)$ is a *one-dimensional* Markov process. The transition density function for $a(t)$, $p(a|a_0;t)$, is governed by the following FPK equation.

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial a}\left[\left\{ \frac{\epsilon^2 F(a)}{\omega_0} - \frac{\pi S_f(\omega_0)}{2a\omega_0^2} \right\}p\right] + \frac{\pi S_f(\omega_0)}{2\omega_0^2}\frac{\partial^2 p}{\partial a^2} \qquad (103)$$

Equations (102) and (103) must be solved subject to the following initial conditions: as $t \to 0$

$$p(a|a_0;t) \rightarrow \delta(a-a_0) \tag{104}$$

and

$$p(a,\phi|a_0,\phi_0;t) \rightarrow \delta(a-a_0)\,\delta(\phi-\phi_0) \tag{105}$$

It is noted that, in the case where $\epsilon^2 h(x,\dot{x})$ is a linear damping term, $\epsilon^2 h(x,\dot{x})$ may be written, as before, in the form of equation (33). One then has $F(a) = \zeta\omega_0^2 a$ (see equation (35)) and $G(a) = 0$. More generally, if $h(x,\dot{x})$ is a function of x only then, from equations (29) and (30), it follows that $F(a) = 0$. Similarly, if $h(x,\dot{x})$ depends on $\dot{x}$ only, then $G(a) = 0$.

2.7 Stationary solutions

Since the excitation is assumed here to be stationary, the response will approach stationarity as time elapses - i.e.

$$\lim_{t\rightarrow\infty} p(a,\phi|a_0,\phi_0;t) = w(a,\phi) \tag{106}$$

where $w(a,\phi)$ is the stationary density function for $a(t)$ and $\phi(t)$. Similarly

$$\lim_{t\rightarrow\infty} p(a,a_0;t) \rightarrow w(a) \tag{107}$$

where $w(a)$ is the stationary density function for $a(t)$.

Equations for $w(a,\phi)$ and $w(a)$ may be obtained by setting $\partial p/\partial t = 0$ in equations (102) and (103). Hence, for $w(a,\phi)$,

$$0 = \frac{\partial}{\partial a}\left[\left\{ \frac{\epsilon^2 F(a)}{\omega_0} - \frac{\pi S_f(\omega_0)}{2a\omega_0^2} \right\} w\right] + \frac{\epsilon^2 G(a)}{a\omega_0} \frac{\partial w}{\partial \phi}$$

$$+ \frac{\pi S_f(\omega_0)}{2a\omega_0^2} \left[\frac{\partial^2 w}{\partial a^2} + \frac{1}{a^2} \frac{\partial^2 w}{\partial \phi^2} \right] \qquad (108)$$

and for $w(a)$

$$0 = \frac{d}{da}\left[\left\{ \frac{\epsilon^2 F(a)}{\omega_0} - \frac{\pi S_f(\omega_0)}{2a\omega_0^2} \right\} w\right] + \frac{\pi S_f(\omega_0)}{2\omega_0^2} \frac{d^2 w}{da^2} \qquad (109)$$

A comparison of equations (108) and (109) reveals that the stationary density function, $w(a,\phi)$, must be related to $w(a)$ as follows

$$w(a,\phi) = \frac{1}{2\pi} w(a) \qquad (110)$$

Here the factor $1/2\pi$ arises from the normalisation condition

$$\int_0^\infty w(a)da = \int_0^{2\pi} \int_0^\infty w(a,\phi)da d\phi = 1 \qquad (111)$$

Equation (110) shows that, when stationarity is reached, the phase angle ϕ is uniformly distributed between 0 and 2π.

Through a transformation from a,ϕ variables to the original $x,\dot{x}$ variables, an expression for the stationary density function of x and $\dot{x}$, $w(x,\dot{x})$ can be deduced from equation (110). Thus

$$w(x,\dot{x}) = \frac{1}{2\pi\omega_0 a} w(a) \qquad (112)$$

where

$$a = \left[x^2 + \frac{\dot{x}^2}{\omega_0^2} \right]^{\frac{1}{2}} \tag{113}$$

and $w(a)$ is the solution of equation (109). This solution is readily found to be (see equation (1.72))

$$w(a) = Ca \exp\left\{ - \frac{2\epsilon^2 \omega_0}{\pi S_f(\omega_0)} \int_0^a F(\xi) d\xi \right\} \tag{114}$$

where C is a normalisation constant.

In the linear case ($\epsilon^2 h(x,\dot{x}) = 2\zeta\omega_0 \dot{x}$) , equations (112) and (114) lead to the following results:

$$w(a) = \frac{a}{\sigma^2} \exp\left\{ - \frac{a^2}{2\sigma^2} \right\} \tag{115}$$

and

$$w(x,\dot{x}) = \frac{1}{2\pi\omega_0\sigma^2} \exp\left\{ - \frac{1}{2\sigma^2} \left[x^2 + \frac{\dot{x}^2}{\omega_0^2} \right] \right\} \tag{116}$$

where

$$\sigma^2 = \frac{\pi S_f(\omega_0)}{2\zeta\omega_0^3} \tag{117}$$

An appropriate integration involving $w(x,\dot{x})$, as given by equation (116), reveals that σ , as given by equation (117) is the stationary standard deviation of x . In the special case of white noise excitation, where $S_f(\omega) = S_0$, a constant, equations (115) to (117) agree with the well-known exact results for this case (Crandall and Mark (1964)). According to equation (115), $a(t)$ has a Rayleigh distribution whereas, from equation (116), the joint distribution of x and $\dot{x}$ is Gaussian.

It is also of interest to compare the above approximate solutions with known exact solutions for non-linear oscillators excited by white noise. One such exact solution exists for the case where the non-linearity is of the form

$$\epsilon^2 h(x,\dot{x}) = 2\zeta\omega_0\dot{x} + \epsilon^2 q(x) \tag{118}$$

(see equations (1.78) to (1.81)). In this case one finds that

$$\epsilon^2 F(a) = a\zeta\omega_0^2 \tag{119}$$

just as in the case of linear stiffness - i.e. equations (115) to (117) still hold. Thus, according to the stochastic averaging theory, the small nonlinear stiffness term does not contribute to the stationary response distribution. However, as one would expect, the exact solution for white noise excitation (see equation (1.79)) shows that the non-linearity in stiffness can markedly affect the distribution of the response. This apparent anomaly is easily resolved when it is recalled that the stochastic averaging solution is an approximation which is only correct to order ϵ^2. If the exact solution is expanded in powers of ϵ^2, and terms of order higher than ϵ^2 are discarded, then one finds that the non-linear stiffness does indeed vanish. Thus the stochastic averaging approximation is consistent with the exact solution.

As noted earlier, in Chapter One, an exact solution exists for a special type of non-linear damping (see equations (1.82) to (1.83)) when the stiffness is linear, the appropriate form of damping is

$$\epsilon\, h(x,\dot{x}) = \epsilon^2\, \rho(a)\dot{x} \tag{120}$$

where $\rho(a)$ is some arbitrary function of a. For this kind of damping one finds, from equation (29), that

$$F(a) = \frac{\omega_0 a \rho(a)}{2} \tag{121}$$

Hence, from equations (116) and (117),

$$w(x,\dot{x}) = \frac{C}{2\pi\omega_0} \exp\left\{ - \frac{\epsilon^2 \omega_0^2}{\pi S_f(\omega_0)} \int_0^a \rho(a)da \right\} \tag{122}$$

This is in complete agreement with the exact solution (see equation (1.83)).

The principal advantage of the approximate solutions given by equations (115) to (117) is that they give explicit results for general types of non-linear damping, where exact results do not exist, even for the case of white noise excitation. Various useful statistics can be computed from $w(x,\dot{x})$, as given by equation (116), including level crossing rates. Specific results have been obtained for the following types of non-linear damping: linear-plus-power law (Roberts (1977)), van der Pol (Spanos (1978,1980)), Rayleigh (Spanos (1978)), and Coulomb (Brouwers (1982)). Generally the results are in very good agreement with corresponding digital simulation results, when the damping is light.

2.8 Linear-plus-power law damping

Suppose that a specific form of non-linear damping is now considered - i.e.

$$h(x,\dot{x}) = \dot{x}(1 + \lambda|\dot{x}|^n) \tag{123}$$

so that the equation of motion is

$$\ddot{x} + \epsilon^2\dot{x}(1 + \lambda|\dot{x}|^n) + \omega_0^2 x = \epsilon\, z(t) \tag{124}$$

It has been shown, earlier, that this form of damping can be used to model the dissipation forces arising from the internal friction of metals.

From equations (29) and (30) it is found that, for this type of damping, one has

$$F(a) = \frac{1}{2} a\omega_0 + \frac{2\lambda}{\pi}(a\omega_0)^{n+1} I_{n+2} \tag{125}$$

where

$$I_n = \int_0^{\pi/2} \cos^n\theta d\theta \tag{126}$$

Hence, from equations (112) and (114), one obtains an expression for the stationary joint density function of the non-dimensional variables

$$x = \frac{x(t)}{\sigma_0} \quad , \quad y = \frac{\dot{x}(t)}{\sigma_0\omega_0}$$

where σ_0 is the standard deviation of the response in the linear case ($\lambda = 0$) ; from equation (117), noting that here $\epsilon^2 = 2\zeta\omega_0$, one has

$$\sigma_0^2 = \frac{\pi S_f(\omega_0)}{\epsilon^2\omega_0^2} \tag{127}$$

in agreement with linear theory (see equation (7)). The result is (see also Roberts (1977))

$$w(x,y) = C\, w_0(x,y)\, \exp\left\{ - \alpha_n\lambda^* \left[\frac{1}{2}(x^2+y^2)\right]^{\frac{n+2}{2}} \right\} \tag{128}$$

where

$$\lambda^* = \lambda\omega_0^n\sigma_0^n \tag{128}$$

$$w_0(x,y) = \frac{1}{2\pi}\exp[-\frac{1}{2}(x^2+y^2)] \tag{129}$$

and

$$\alpha_n = [8(2)^{n/2}/(n+2)\pi]I_{n+2} \tag{130}$$

C is a normalisation constant, such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)dxdy = 1 \tag{131}$$

It is noted that λ^* is a non-dimensional non-linearity parameter and $w_0(x,y)$ is the (Gaussian) result when $\lambda^* = 1$ (noting that $C = 1$ when $\epsilon^* = 0$) C may be expressed as

$$C = \left[\int_0^{\infty}\exp[-Q(v)]dv\right]^{-1} \tag{132}$$

where

$$Q(v) = v + \alpha_n\lambda^* v^{\frac{n+2}{2}} \tag{133}$$

From the basic result given by equation (128) one can evaluate various statistics of the response process, x(t) . For example, the moments of the response are, correct to order λ^* , given by

$$E\left\{\left[\frac{x(t)}{\sigma_0}\right]^m\right\} = \frac{I_m 2^{\frac{m+2}{2}}}{\pi}\Gamma(\frac{m}{2}+1)[1-\lambda^* A_{nm}] + O(\lambda^{*2}) \tag{134}$$

where

$$A_{nm} = \alpha_n \left\{ \frac{\Gamma\left[\frac{m+n+4}{2}\right]}{\Gamma\left[\frac{m+2}{2}\right]} - \Gamma\left[\frac{n+4}{2}\right] \right\} \qquad (135)$$

When $m = 2$, equation (134) gives the mean square response, $\sigma^2 = E\{x^2(t)\}$. In this case

$$\frac{\sigma^2}{\sigma_0^2} = 1 - \lambda^*[(n+1)(n-1)\ldots(1)] + O(\lambda^{*2}) \qquad (136)$$

if n is even, and

$$\frac{\sigma^2}{\sigma_0^2} = 1 - \lambda^*\sqrt{\frac{2}{\pi}}\,[(n+1)(n-1)\ldots(2)] + O(\lambda^{*2}) \qquad (137)$$

if n is odd. These results are identical to those obtained by Crandall et al. (1964), using the perturbation method.

The above result for the mean square response can also be compared with that obtained by the equivalent linearisation method. If equation (124) is replaced by

$$\ddot{x} + \beta_{eq}\dot{x} + \omega_0^2 x = \epsilon\, z(t) \qquad (138)$$

then β_{eq} can be chosen to minimise the mean square of the difference between the two equations. The optimum value of β_{eq} is

$$\beta_{eq} = \epsilon^2\left[\, 1 + \lambda\, \frac{E\{|\dot{x}|^{n+2}\}}{E\{\dot{x}^2\}} \,\right] \qquad (139)$$

Assuming a Gaussian distribution for $\dot{x}$, with standard deviation σ', equation (139) leads to the result

$$\beta_{eq} = \epsilon^2\left[\, 1 + \lambda\sigma'^n \left[\frac{2^{n+2}}{\pi}\right]^{\frac{1}{2}} \Gamma\left[\frac{n+3}{2}\right] \,\right] \qquad (140)$$

For the linearised oscillator

$$\sigma' = \omega_0 \sigma = \omega_0 \sigma_0 \left[\frac{\beta}{\beta_{eq}}\right] \tag{141}$$

and equations (140) and (141) may be solved iteratively to find β_{eq}, and hence σ^2.

Fig. 2.4 shows the variation of σ^2/σ_0^2 with λ^*, as predicted by the stochastic averaging method (denoted SA theory), for $n = 2$. For this value of n the mean square response can be evaluated analytically (Roberts (1977)). The result is

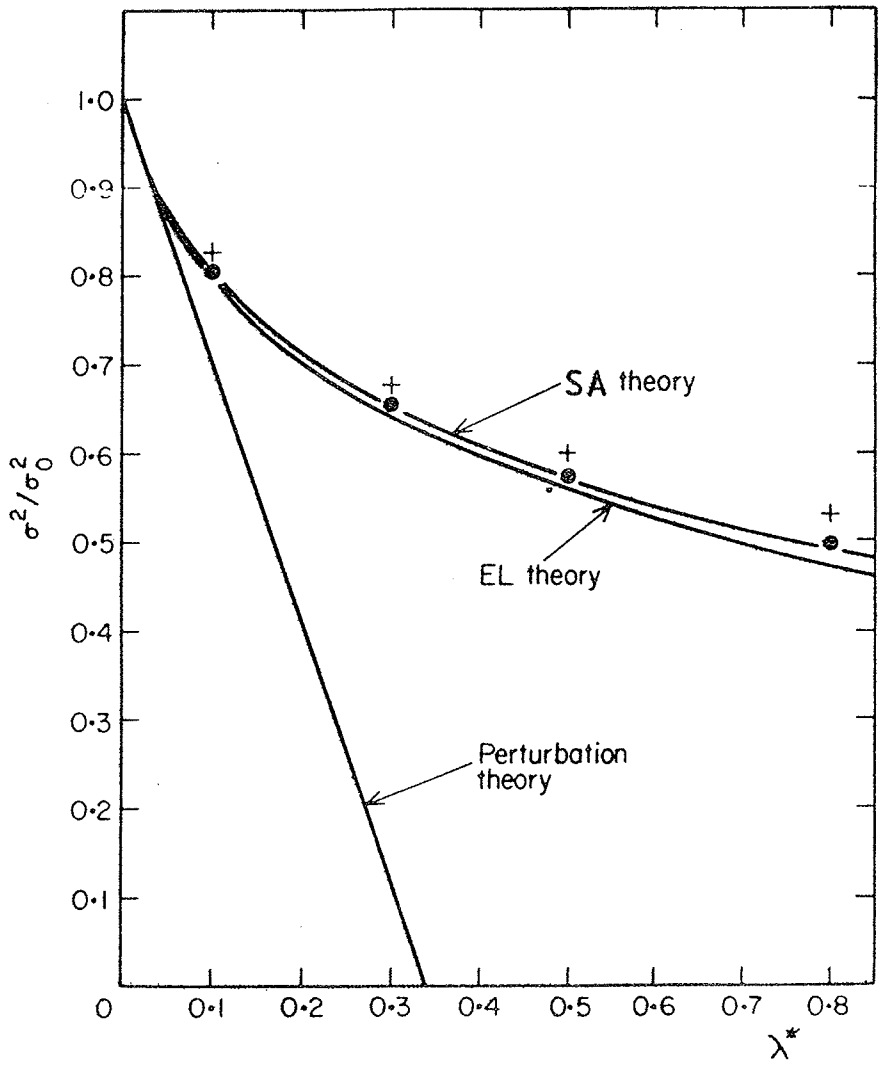


Fig. 2.4.

$$\frac{\sigma^2}{\sigma_0^2} = 2\theta\left\{ \left[\sqrt{\pi}\, e^{\theta^2}(1 - \mathrm{erf}\theta)\right]^{-1} - \theta \right\} \qquad (142)$$

where

$$\theta = 1/(3\lambda^*)^{\frac{1}{2}} \qquad (143)$$

Also shown in Fig. 2.4 is the corresponding perturbation solution

$$\frac{\sigma^2}{\sigma_0^2} = 1 - 3\lambda^* \qquad (144)$$

(see equation (136)). For $n = 2$ equations (140) and (141) can be solved explicitly to yield the equivalent linearisation solution (EL Theory)

$$\frac{\sigma^2}{\sigma_0^2} = \left[\sqrt{1+12\lambda^*} - 1\right] / 6\lambda^* \qquad (145)$$

Fig. 2.4 shows that the EL solution is very close to the SA result, over the range of λ^* shown. It is noted that both the SA and the EL results agree with the perturbation solution when λ^* is small, to order λ^*.

Digital simulation estimates, also shown in Fig. 2.4, are seen to be in excellent agreement with the SA theory. As one would expect from the approximations inherent in the SA theory, the agreement is less satisfactory for the higher damping factor, (noting that $\epsilon^2 = 2\zeta\omega_0$) although the discrepancy is quite small.

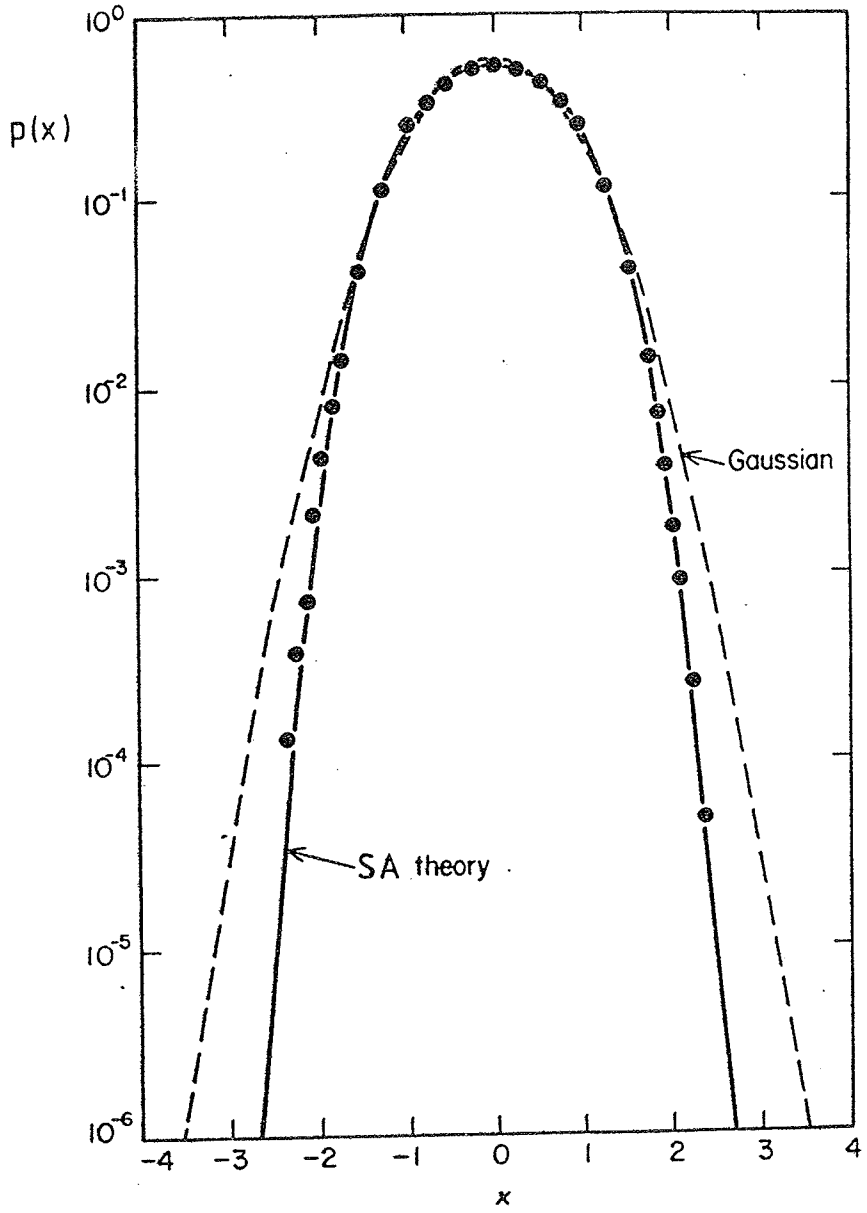


Fig. 2.5.

Fig. 2.5 shows a typical result for the probability density functions of $x(t)$, obtained by integrating $w(x,y)$, as given by equation (128). Here $\lambda^* = 1$ and $n = 2$. The full line shows the variation of $w(x)$ with x, according to stochastic averaging, whereas the broken line shows the corresponding Gaussian density function, with the same mean-square value. It is observed that there is a very pronounced deviation from the Gaussian form

in the tails of the distribution. Simulation results are seen to be in excellent agreement with the stochastic averaging result.

The expected frequency of up-crossings, ν of some fixed level, a , can also be calculated from a knowledge of w(x,y) . One has Rice's formula

$$\nu = \int_0^\infty y w(\eta, y)\, dy \tag{146}$$

where $\eta = a/\sigma_0$ is a non-dimensional amplitude. For n = 2 an analytical expression for ν may be found

$$\frac{\nu}{\nu_0} = \exp\left[\frac{\eta^2}{2}\right] \frac{\left[1 - \exp\left[\theta + \frac{\eta^2}{4\theta}\right]\right]}{[1 - \exp\theta]} \tag{147}$$

where ν_0 is the linear result $(\lambda^* = 0)$. When $\eta = 0$ this result reduces to

$$\nu = \nu_0 = \frac{\omega_0}{2\pi} \tag{148}$$

Fig. 2.6 shows the computed variation of ν/ν_0 with η , for n = 2 , and $\lambda^* = 0.05$, 0.10 and 0.20 , according to stochastic averaging. The simulation estimates of ν , shown in this figure, are in very good agreement with the SA theory.

3. Oscillators with parametric excitation

Random parametric excitation frequently occurs in engineering vibration problems. It has been demonstrated by many authors, following Stratonovich's pioneering work, that the averaging method is particularly effective in assessing the stability of systems with this type of excitation.

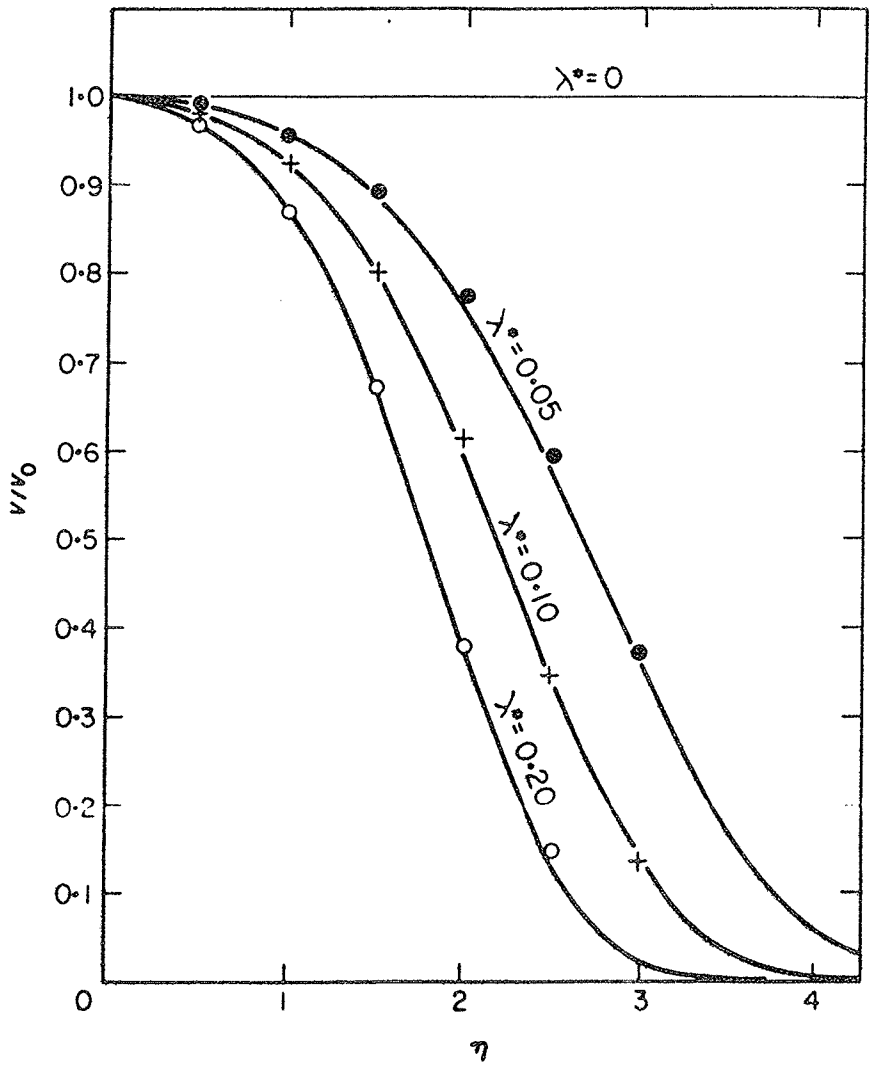


Fig. 2.6.

Here the case of an oscillator with non-linear damping and a random, parametric term will be used to illustrate the method of deriving stability criteria.

3.1 Equations of motion

Consider an oscillator with the following equation of motion:

$$\ddot{x} + \epsilon^2 h(x,\dot{x}) + \omega_0^2 x[1 + \epsilon q(t)] = \epsilon z(t) \qquad (149)$$

Here $q(t)$ is the parametric excitation, other terms retaining their previous meaning. The non-parametric excitation is scaled, as before, by ensure that the response magnitude is of order ϵ^0. It can be shown (e.g. Roberts (1982)) that the parametric excitation must be similarly scaled, as above, to ensure the response remains of order ϵ^0.

Introducing the van der Pol transformation, as before, (see equations (22) and (23)) one can express the equation of motion as a pair of first order, coupled equations for a and ϕ. Hence, applying the Stratonovich-Khasminskii limit theorem one obtains the following "simplified" Itô equations:

$$da = (\epsilon^2 F_1(a) + \frac{3\alpha}{2}a + \frac{\beta}{2a})dt + (\beta+\alpha a^2)^{\frac{1}{2}}dW_1 \qquad (150)$$

$$d\phi = \gamma dt + (\frac{\beta}{a^2} + \alpha + \delta)^{\frac{1}{2}}dW_2 \qquad (151)$$

where

$$\alpha = \pi S_p(2\omega_0)/4\omega_0^2 \qquad (152)$$

$$\beta = \pi S_f(\omega_0)/\omega_0^2 \qquad (153)$$

$$\gamma = \frac{1}{4\omega_0^2}\int_0^\infty w_p(\tau)\sin 2\omega_0\tau d\tau \qquad (154)$$

$$\delta = \pi S_p(0)/2\omega_0^2 \qquad (155)$$

and $S_p(\omega)$, $S_f(\omega)$ are, respectively, the power spectra of the processes $p(t)$ and $f(t)$, where

$$p(t) = \epsilon q(t) \quad , \quad f(t) = \epsilon z(t) \qquad (156)$$

Similarly, $W_p(\tau)$ is the correlation function of $p(t)$. The function $F_1(a)$ is (apart from a constant factor) equal to the function $F(a)$ defined by equation (29) (here $G(a) = 0$). Thus

$$F_1(a) = \frac{1}{\omega_0} F(a) \tag{157}$$

Clearly when the parametric term is absent, $\alpha = \delta = 0$ and equations (150) and (151) reduce to equations (66) and (67).

3.2 The FPK equations

From equations (150) and (151), describing the asymptotic ($\epsilon^2 \to 0$) Markov model for the joint process $a(t)$, $\phi(t)$, an appropriate FPK equation is easily derived. Generalising (102) one has

$$\frac{\partial p}{\partial a} = -\frac{\partial}{\partial a}\left[\left[\epsilon^2 F_1(a) + \frac{3\alpha}{2}a + \frac{\beta}{2a}\right]p\right]$$

$$-\frac{\partial}{\partial \phi}(\gamma p) + \frac{1}{2}\frac{\partial^2}{\partial a^2}[(\beta + \alpha a^2)p]$$

$$+\frac{1}{2}\frac{\partial^2}{\partial \phi^2}\left[\left[\frac{\beta}{a^2} + \alpha + \delta\right]p\right] \tag{158}$$

for the transition density $p(a,\phi|a_0,\phi_0;t)$. This function must satisfy the initial condition given by equation (105).

As in the non-parametric case, $a(t)$ is uncoupled from $\phi(t)$, as is evident from equation (150). Thus $a(t)$ is a one-dimensional Markov process, with a transition density function governed by

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial a}\left[\left[\epsilon^2 F_1(a) + \frac{3\alpha}{2}a + \frac{\beta}{2a}\right]p\right]$$

$$+\frac{1}{2}\frac{\partial^2}{\partial a^2}[(\beta + \alpha a^2)p] \tag{159}$$

The conditions under which a stationary solution to these equations, as indicated by equations (106) and (107) exists depends on the form of the damping.

3.3 Linear damping

If $\epsilon^2 h(x,\dot{x}) = 2\zeta\omega_0\dot{x}$, then

$$\epsilon^2 F_1(a) = -\zeta\omega_0 a \tag{160}$$

and, with $\partial p/\partial t = 0$, the stationary density function of $a(t)$, $w(a)$, must satisfy (see equation (159))

$$\frac{1}{2}\frac{d^2}{da^2}[(\beta + \alpha a^2)w] - \frac{d}{da}\left[\left[-\zeta\omega_0 a + \frac{3\alpha}{2}a + \frac{\beta}{2a}\right]w\right] = 0 \tag{161}$$

The solution to this equation is

$$w(a) = \frac{2\alpha\lambda\beta^{\lambda}a}{(\beta+\alpha a^2)^{1/2}} \tag{162}$$

where

$$\lambda = \frac{(\zeta\omega_0-\alpha)}{\alpha} \tag{163}$$

This solution will only exist if $\lambda > 0$ - i.e.

$$\zeta > \frac{\pi S_p(2\omega_0)}{4\omega_0^3} \tag{164}$$

It has been shown by Ariaratnam and Tam (1979) that the inequality of equation (163) gives the condition for sample stability for the approximate amplitude process governed by its simplified equation, and they have given some justification for

the assumption that it also gives the condition for sample stability of the original system.

It is interesting to note that the stability criterion stated above is not affected by the presence of the non-parametric excitation process, f(t) . In fact the critical damping factor depends only on the natural frequency of oscillation, ω_0 , and the value of the power spectrum of p(t) , at *twice* the natural frequency. This is not unexpected since it is well known that the stability boundary for a linear system excited by a periodic, parametric input has a minimum when the excitation frequency is twice that the natural frequency. The stability criterion of equation (164) was first deduced by Stratonovich and Romanovskii (1958).

From equation (162) it is a simple matter to evaluate the moments

$$E\{a^n\} = \int_0^\infty w(a)a^n da \tag{165}$$

One finds that

$$E\{a^n\} = \frac{\lambda\beta^{n/2}\Gamma(1+\frac{n}{2})\Gamma(\lambda-\frac{n}{2})}{2^{n/2}\Gamma(\lambda+1)} \tag{166}$$

The n^{th} moment exists if

$$\lambda > n/2 \tag{167}$$

Hence, for example, for stability of the second moment one has

$$\zeta > \frac{\pi S_p(\omega_0)}{2\omega_0^3} \tag{168}$$

Returning to equation (162) it is observed that, when stability exists, the amplitude distribution is of a power-law type, rather than Rayleigh. However, using the standard formula

$$\lim_{\nu \to 0} (1+\nu x)^{\frac{1}{\nu}} = \exp(x) \tag{169}$$

one can show that equation (162) does indeed approach the Rayleigh form as $\alpha \to 0$. Thus

$$w(a) \to \frac{a}{\sigma^2} \exp\left\{-\frac{a^2}{2\sigma^2}\right\} \tag{170}$$

where

$$\sigma^2 = \frac{\pi S_f(\omega_0)}{2\zeta\omega_0^3} \tag{171}$$

As in the non-parametric case, the joint distribution of a and ϕ is given by equation (110), and for x and $\dot{x}$ one has equation (112) again. Hence various distributions, such as that for $x(t)$, can be computed (see Roberts (1982)).

3.4 Non-linear damping

To illustrate the effect of non-linear damping, consider the specific form

$$\epsilon^2 h(x,\dot{x}) = 2\zeta\omega_0\dot{x} + \rho|\dot{x}|\dot{x} \tag{172}$$

This represent linear-plus-quadratic damping, often to be found in fluid-structure interaction problems (see Chapter One).

For this type of damping one finds, from equations (29) and (156) that

$$\epsilon^2 F_1(a) = -\omega_0 a\left[\zeta + \frac{4}{3\pi}\rho a\right] \tag{173}$$

Substituting this result into the FPK equation for $a(t)$ one obtains the stationary solution

$$w(a) = \frac{ca}{(\beta+\alpha a^2)^{1+\lambda}} \exp\left\{-\frac{1}{\mu}\left[a - \left[\frac{\beta}{\alpha}\right]^{\frac{1}{2}} \tan^{-1}\left\{\left[\frac{\beta}{\alpha}\right]^{\frac{1}{2}} a\right\}\right]\right\} \tag{174}$$

where c is a normalisation constant and

$$\mu = \frac{3\pi\alpha}{8\rho\omega_0} \tag{175}$$

3.4.1 Purely parametric excitation

For linear damping it has been seen that the motion is stable if $\lambda > 0$. When this condition is satisfied, the response level is determined by the level of the excitation process. When $f(t)$ is absent, the response is zero.

When non-linear, quadratic damping is included in the theoretical model the motion is still stable for $\lambda > 0$, of course. However, if $\lambda < 0$, it is possible for the non-linear damping to limit the amplitude of motion - in other words, it is possible for the roll motion to reach a stationary condition, with a non-zero standard deviation. This is analogous to the phenomenon of limit cycling, which is well known in deterministic non-linear vibrations. In physical terms the motion, which is unstable for $\lambda < 0$, (according to linear theory) will build up until the mean rate of energy dissipation due to damping is equal to the mean rate of energy input from the parametric excitation.

If $f(t)$ is assumed to be zero it follows that $\beta = 0$. Hence the stationary density function, given by equation (174), reduces to

$$w(a) = \frac{c}{\alpha^{1+\lambda} a^{1+2\lambda}} \exp\left[-\frac{a}{\mu}\right] \qquad (176)$$

where the normalisation constant is given by

$$c = \mu^{2\lambda} \alpha^{1+\lambda} \Gamma(-2\lambda) \qquad (177)$$

This solution is valid for $\lambda < 0$.

The stationary joint density function for $a(t)$ and $\phi(t)$ is again given by equation (110) and, through integration, various response statistics can be found. For example (see Roberts (1982))

$$\sigma^2 = \lambda(2\lambda-1)\mu^2 \qquad (\lambda < 0) \qquad (178)$$

The analysis clearly shows that linear and quadratic components of damping have different effects, so far as stability is concerned. In the absence of the excitation process, $f(t)$, there is a critical value of ζ above which the motion is stable, when the damping is linear. When non-linear damping is introduced the motion is stable for all possible values of λ , from 0 to -1 (see equation (162)).

3.4.2 Combined excitation

From equations (112) and (174), the joint density of x and $\dot{x}$ can be found, in the case where both $f(t)$ and $p(t)$ are present. Hence one can obtain statistics such as the mean-square response. Analysis is not easy in the case of linear-plus-quadratic damping, since the required integrals are not easy to evaluate analytically. Explicit results can, however, be found in the case of linear-plus-cubic damping (Roberts (1982)).

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AVERAGING METHODS IN RANDOM VIBRATION

J. B. Roberts

CHAPTER THREE

Generalisations and Extensions of Stochastic Averaging

1. Introduction

In the preceeding Chapter the "standard" method of stochastic averaging was introduced and applied to the case of an oscillator with non-linear damping; the cases of purely non-parametric excitation and purely parametric excitation were considered in some detail, and explicit results obtained for some specific cases.

In this Chapter it will be shown, initially, that the standard averaging method has a much wider range of application. Firstly it will be demonstrated that it is possible to analyse multi-degree of freedom systems, with parametric excitation, and to derive stability criteria (at least in the linear case). Combined external and parametric excitation may also be considered with this approach; unfortunately, in many cases the FPK equations which result from an application of the limit theorem are very difficult to solve analytically. Secondly, it will be shown that it is possible to allow for non-stationarity in the excitation, without a great increase in complexity. Again the FPK equations become more difficult to solve but results can be obtained fairly readily for oscillators with non-linear damping, by semi-analytical, or wholly numerical methods.

As pointed out in Chapter Two, a disadvantage of the standard averaging method is that it does not enable one to

examine the influence of non-linearity in stiffness on the probability distribution of the response. In the second half of this Chapter it will be demonstrated that this difficulty can be overcome completely, in the case of single degree of freedom systems, by using the energy envelope process, rather than the amplitude process, as the basis for the averaging procedure.

2. Multi-degree of freedom systems

Applications of stochastic averaging to multi-degree of freedom systems have been mainly concerned with situations where parametric excitation is present. Assessment of stability is then of major concern.

A fairly general form for the equations of motion of a linear system, with purely parametric excitation, and n degrees of freedom, is as follows:

$$\ddot{x}_i + \omega_i^2 x_i + \epsilon^2 \sum_{j=1}^{n} \beta_{ij}\dot{x}_i + \epsilon q(t) \sum_{j=1}^{n} k_{ij} x_i = 0 \qquad (1)$$

$$(i = 1,2,\ldots,n)$$

Here x_i are principal coordinates and the coefficients β_{ij}, k_{ij} are constants; ω_i are the natural frequencies of the undamped system, with no excitation.

The application of the stochastic averaging method in this case is a natural extension of that used for single-degree of freedom systems (see Chapter Two). Thus, new variables, $a_i(t), \phi_i(t)$ are introduced by means of the transformations

$$x_i(t) = a_i(t)\cos\Phi_i(t) \qquad (2)$$

$$\dot{x}_i(t) = -a_i(t)\omega_i\sin\Phi_i(t) \qquad (3)$$

where

$$\Phi_i = \omega_i t + \phi_i \tag{4}$$

The equations of motion, when cast in terms of $a_i(t)$,$\phi_i(t)$ variables, are of the standard form required by the Stratonovich-Khasminskii limit theorem (see section 2.4). Hence, applying this theorem, a set of Itô equations for a_i,ϕ_i can be derived. These are of the form

$$da_i = \epsilon^2 m_i(a) + \epsilon \sum_{j=1}^{n} \sigma_{ij} dW_j^a \tag{5}$$

$$d\phi_j = \epsilon^2 n_i(a) + \epsilon \sum_{j=1}^{n} \mu_{ij} dW_j^\phi \tag{6}$$

Here W_j^a, W_j^ϕ $(j = 1,2...,n)$ are independent unit Wiener processes and m_i, n_i and σ_{ij}, μ_{ij} $(i,j = 1,2,...,n)$ are drift and diffusion coefficients, respectively.

It is found that the equations for amplitude are uncoupled from those of phase and hence $a_i(t)$ $(i = 1,2,...,n)$ is a diffusive n-vector Markov process, with an appropriate FPK equation. The problem of solving the FPK equation can be circumvented by using the Itô differentiation rule (Sobczyk (1985[1])) to derive a set of differential equations for the moments of a_i . Thus, the second moments

$$M_i = E\{a_i^2(t)\} \tag{7}$$

are governed by differential equations of the form

$$\frac{dM_i}{dt} = \epsilon^2 \sum_{j=1}^{n} A_{ij} M_i \tag{8}$$

where the coefficients A_{ij} can be expressed in terms of the drift and diffusion coefficients. A necessary and sufficient condition for second moment stability is that all the eigenvalues of the coefficient matrix $\underset{\sim}{A} = [a_{ij}]$ have negative real parts. For specific systems, this enables stability conditions to be formulated fairly easily, using the Routh-Hurwitz criteria.

The results of this approach show that second moment stability depends only upon the values of the excitation spectral density near twice the natural frequency, and the combination frequencies $|\omega_i \pm \omega_j|$ - i.e.

$$\begin{array}{lll} S_q(2\omega_i) & i = 1,2,\ldots,n & \\ S_q(|\omega_i \pm \omega_j|) & i,j = 1,2,\ldots,n \qquad i \neq j & (9) \end{array}$$

This method of analysis can be extended to include situations where external, non-parametric excitation is also present (Dimentberg (1980,1983(a), 1983(b))). For example, Dimentberg (1983(b)) has analysed the case of a pair of coupled oscillators, with combined external and parametric excitation. It is also a fairly simple matter to consider the effect of harmonically varying parametric excitation (Dimentberg and Isikov (1977), Ariaratnam and Tam (1977)).

Whilst it is relatively easy to obtain expressions for the moments of the response, it is much more difficult to obtain results for the probability density functions. Some limited results, for specific systems, are given by Dimentberg (1983(b)) and Dimentberg and Isikov (1977). As in the case of a simple oscillator with parametric excitation (see Chapter Two), these density functions are of the power-law type.

For non-linear multi-degree of freedom systems very limited results exist at present. One principal difficulty here is that the moment equations form an infinite hierachy, which requires the imposition of some closure scheme to obtain a solution (see Chapter One). Some results for the probability density functions

of two degree of freedom systems have been reported by Schmidt (1977) and Schmidt and Schultz (1983).

2.1 Applications

Nemat-Nasser (1972) and Ariaratnam and Srikantiah (1978) have analysed the flexural-torsional stability of a simply supported beam subjected to stochastically varying end couples. By considering the fundamental modes of bending and torsion the equations of motion can be written in the form of a pair of coupled, ordinary differential equations, of the form of equation (1). Similarly, the flexural-torsional stability of a beam subjected to a transverse load at its mid-span can be analysed (Ariaratnam and Tam (1977)). Other two degree of freedom structural systems have been considered (Ariaratnam and Tam (1973)) including a gyroscopic system consisting of a rotating shaft, with unequal flexural rigidities, subjected to a randomly varying axial thrust (Ariaratnam (1972)).

Coupled motion arising from fluid-structure interaction has also been investigated by Lin and Holmes (1978), in connection with wind loading on structures, and by Fujimori et al. (1979) and Prussing and Lin (1982,1983), in connection with the coupled flapping-torsion and flapping-lag motion of helicopter rotor blades.

3. Non-stationary excitation and response

If the excitation process, f(t) say, is *non-stationary*, and with zero-mean, it can be described in terms of an "evolutionary power spectrum", $S(\omega,t)$. The evolutionary spectrum can be expressed in the form

$$S_f(\omega,t) = |A(t,\omega)|^2 \, \bar{S}_f(\omega) \qquad (10)$$

where $A(t,\omega)$ and $\bar{S}_f(\omega)$ are appropriate functions and the symbol $|\ |$ denotes the modulus of the complex function (see Priestley (1967)).

In the special case where the process $w(t)$ is stationary,

$$S_f(\omega,t) = S_f(\omega) \tag{11}$$

independent of t, where $S_f(\omega)$ is the usual power spectrum. If

$$f(t) = \alpha(t)n(t) \tag{12}$$

where $\alpha(t)$ is a deterministic modulating function and $n(t)$ is a stationary random process, with power spectrum $S_n(\omega)$, then it can be shown that

$$S_f(\omega,t) = \alpha^2(t)S_n(\omega) \tag{13}$$

This is obviously a generalisation of equation (11) (here $\alpha = 1$). If $n(t)$ is a white noise process then

$$S_f(\omega,t) = \alpha^2(t)S_0 \tag{14}$$

where S_0 is a constant spectral level.

3.1 Non-stationary excitation of oscillators

Consider again the simple oscillator governed by

$$\ddot{x} + \epsilon^2 h(x,\dot{x}) + \omega_0^2 x = \epsilon z(t) \tag{15}$$

where $f(t) = \epsilon z(t)$, as before, but now $z(t)$ is non-stationary, with power spectrum $S_z(\omega,t)$.

Applying the usual stochastic averaging theory one finds the following Itô equations for the amplitude process, $a(t)$, and

phase process, $\phi(t)$

$$da = -\frac{\epsilon^2}{\omega_0} F(a)dt + \frac{\pi S_f(\omega_0,t)}{2a\omega_0^2} dt + \frac{[\pi S_f(\omega_0,t)]^{\frac{1}{2}}}{\omega_0} dW_1(t) \tag{16}$$

$$d\phi = -\frac{\epsilon^2}{a\omega_0} G(a)dt + \frac{[\pi S_f(\omega_0,t)]^{\frac{1}{2}}}{a\omega_0} dW_2(t) \tag{17}$$

In the case of stationary excitation

$$S_f(\omega_0,t) = S_f(\omega_0) \tag{18}$$

and equations (16) and (17) reduce to the equations given earlier, in Chapter 2 (see (2.66) and (2.67)). Hence, the FPK equations for a,ϕ can be obtained from (2.102) and (2.103) by simply replacing $S_f(\omega_0)$ by $S_f(\omega_0,t)$.

3.2 Non-stationary exact solutions

A complete solution to the FPK equation for $a(t)$ is available, so far, only for the linear case. It can be shown (see Spanos (1983), Spanos and Solomos (1983) and Solomos and Spanos (1984) that the transition density function for $a(t)$ is given by

$$p(a,t|a_1,t_1)$$

$$= \frac{a}{c(t_1,t)} \exp\left\{ -\frac{a^2+a_1^2 e^{-2\zeta\omega_0\tau}}{2c(t_1,t)} \right\} I_0\left\{ \frac{aa_1 e^{-\zeta\omega_0 t}}{c(t_1,t)} \right\} \tag{19}$$

Here

$$c(t_1,t) = \frac{\pi}{\omega_0^2} \exp(-2\zeta\omega_0 t) \int_{t_1}^{t} \exp(2\zeta\omega_0\tau')S(\omega_0,\tau')d\tau'$$

and the symbol I_0 denotes the modified Bessel function, of zero order. $\tau = t-t_1$.

Equation (19) can be written in the form

$$p(a,t|a_1,t) = \sum_{n=0}^{\infty} A_n\left[\frac{a^2}{2c}\right] L_n\left[\frac{a_1^2}{2c_1}\right] \rho^{2n} \qquad (20)$$

where

$$c_1 \equiv c(t_1,0) \quad , \quad c \equiv c(0,t) \qquad (21)$$

and

$$\rho^2 = \frac{c_1}{c} \exp(-2\zeta\omega_0\tau) \qquad (22)$$

Here $\rho^2 \leq 1$ and is a measure of the correlation of the process a^2 . In equation (20), L_n denotes the Laguerre polynomial of order n , defined by the equation

$$L_n = \frac{1}{n!} e^z \frac{d^n}{dz^n}\left[e^{-z}z^n\right] \qquad (23)$$

and the function A_n is defined by

$$A_n\left[\frac{a^2}{2c}\right] = \frac{a}{c} \exp\left[-\frac{a^2}{2c}\right] L_n\left[\frac{a^2}{2c}\right] \qquad (24)$$

The transition density function can be used to derive the joint and marginal probability density functions. For example, with $a_1 = a_0$ at $t = t_1 = 0$, the distribution of a at time t is given by the density function

$$p(a,t) = \frac{a}{c}\exp\left\{\frac{-a^2+a_0^2e^{-2\zeta\omega_0 t}}{2c}\right\} I_0\left\{\frac{aa_0e^{-\zeta\omega_0 t}}{c}\right\} \qquad (25)$$

Expressions can also be obtained for the transition density of the joint process $a(t),\phi(t)$, by solving the FPK equation for this process. The result is (Spanos (1983), Solomos and Spanos (1984))

$$p(a,\phi,t|a_1,\phi_1,t_1) = \frac{a}{2\pi c(t_1,t)}\exp\left[-\left\{\frac{\alpha_c^2+\alpha_s^2}{2c(t_1,t)}\right\}\right] \qquad (26)$$

where

$$\alpha_c = a\cos(\omega_0 t+\phi) - a_1\cos(\omega_0 t_1+\phi_1+\omega_0\tau)e^{-\zeta\omega_0\tau} \qquad (27)$$

$$\alpha_s = a\sin(\omega_0 t+\phi) - a_1\sin(\omega_0 t_1+\phi_1+\omega_0\tau)e^{-\zeta\omega_0\tau} \qquad (28)$$

Furthermore, by using relationships between a,ϕ,x and $\dot{x}$, the joint density for $x(t)$ and the scaled velocity $y(t) = \dot{x}/\omega_0$ is found to be

$$p(x,y,t|x_1,y_1,t_1) = \frac{1}{2\pi c(t_1,t)}\exp\left[-\left\{\frac{\alpha_x^2+\alpha_y^2}{2c(t_1,t)}\right\}\right] \qquad (29)$$

where

$$\alpha_x = x - e^{-\zeta\omega_0\tau}(x_1\cos\omega_0\tau + y_1\sin\omega_0\tau) \qquad (30)$$

$$\alpha_y = y - e^{-\zeta\omega_0\tau}(-x_1\sin\omega_0\tau + y_1\cos\omega_0\tau) \qquad (31)$$

It is noted that the variances of both x and y are equal to $c(t_1,t)$, at time t, and that, according to equation (29), the joint distribution of x and y is Gaussian.

One important fact emerges fairly readily from equation (26). If the system is initially at rest ($a_1 = 0$) then a and ϕ are statistically independent. Moreover, the phase distribution is uniform. Hence the marginal density function for (a,ϕ) is given by

$$p(a,\phi,t) = \frac{1}{2\pi} p(a,t) \tag{32}$$

where $p(a,t)$ is given by equation (25). This is a generalisation of the result given earlier, in Chapter Two, for the stationary case.

3.3 Approximate analytical solutions

For non-linear oscillators the general analytical solution of the FPK equations for $a(t)$ alone, and $a(t),\phi(t)$ combined, appears to be difficult. Work in this area has concentrated on determining the marginal density function $p(a,t)$, for the case where $a_1 = 0$ at $t = 0$. Equation (32) then still holds and so $p(a,\phi,t)$ may be determined from a knowledge of $p(a,t)$ only.

One approach is to use a Galerkin technique in which the basis functions are selected by using the corresponding solution for the linear oscillator. Thus, refering to equation (20), the following expansion may be introduced (see Spanos 1981(a) and (b))

$$p(\bar{a},t) = \sum_{n=1}^{N} F_n(t)\, A_n\left[\frac{\bar{a}^2}{2}\right] \tag{33}$$

where

$$\bar{a} = a/S^* \tag{34}$$

and S^* is a scaling constant. N is an appropriate integer and $F_n(t)$ are functions of time, to be determined. Substituting

equation (33) into the FPK equation for $a(t)$, and taking into account the following orthogonality property of the $A_i(\ )$ functions,

$$\int_0^\infty \frac{A_i A_j}{\bar{A}_0}\, d\bar{a} \;=\; \delta_{ij} \tag{35}$$

one can derive a set of ordinary differential equations for $F_n(t)$, which may be solved numerically. Alternatively, for small degrees of non-linearity, they can be determined analytically, using a perturbation technique.

3.4 Random walk numerical method

The FPK equation governing $a(t)$, in the non-stationary case, can be written in the standard form

$$\frac{\partial p}{\partial t} \;=\; -\frac{\partial}{\partial a}\Big\{A(a,t)p\Big\} + \frac{1}{2}\frac{\partial^2}{\partial a^2}\Big\{B(a,t)p\Big\} \tag{36}$$

where

$$A(a,t) \;=\; -\frac{\epsilon^2 F(a)}{\omega_0} + \frac{\pi S_f(\omega_0,t)}{2a\omega_0^2} \tag{37}$$

$$B(a,t) \;=\; \frac{\pi S_f(\omega_0,t)}{\omega_0^2} \tag{38}$$

$A(a,t)$ and $B(a,t)$ are related to the "infinitesimal moments" of $a(t)$, defined by

$$\alpha_n(a,t,\Delta t) \;=\; E\{[a(t) - a(t-\Delta t)]^n\} \;=\; E\{\Delta a^n\} \tag{39}$$

where Δa is the change in $a(t)$, during an interval of time Δt . Using the transition density function $p(a,t|a_1,t_1)$, α_n can be expressed as

$$\alpha_n(a,t,\Delta t) = \int_0^\infty d\xi(\xi-a)^n\, p(\xi,t|a,t-\Delta t) \tag{40}$$

From standard Markov process theory one can show that, if $a(t)$ is a Markov process,

$$A(a,t) = \lim_{\Delta t\to 0} \frac{\alpha_1(a,t,\Delta t)}{\Delta t} \tag{41}$$

and

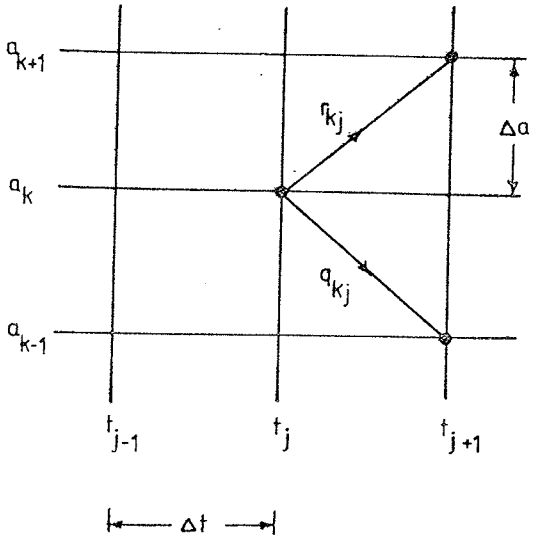
$$B(a,t) = \lim_{\Delta t\to 0} \frac{\alpha_2(a,t,\Delta t)}{\Delta t} \tag{42}$$

whilst

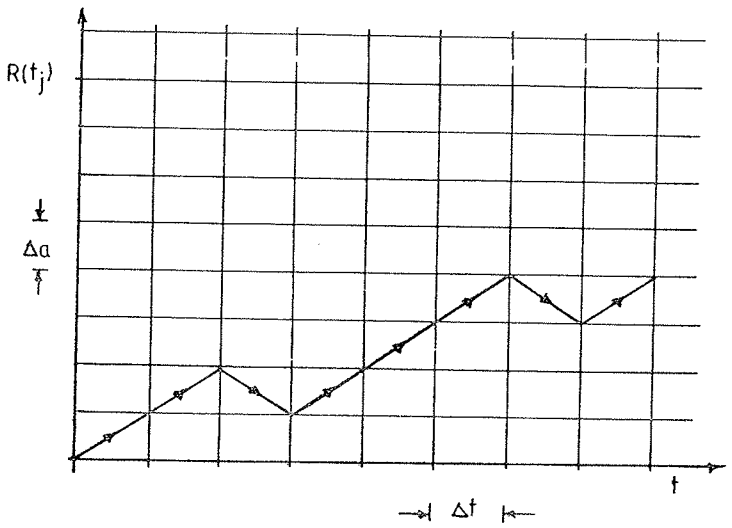
$$\lim_{\Delta t\to 0} \frac{\alpha_n(a,t,\Delta t)}{\Delta t} = 0 \qquad \text{for} \quad n > 2 \tag{43}$$

It is possible to find a variety of discrete random processes which possess the same infinitesimal moments in the limit where the time step approaches zero - i.e. which converge to a continuous diffusion process, described by the FPK equation given by equation (36).

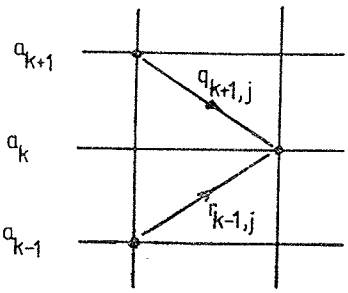
Suppose that $R(t_j)$ is a random walk process, such that it can only assume the discrete amplitudes a_k, where $a_k = k\delta_a$ $(k = 0,1,\ldots,)$. If it is in the state (a_k,t_j) then it will be assumed that it can only move to state (a_{k+1},t_{j+1}) with probability r_{kj}, or to state (a_{k-1},t_{j+1}) with probability $q_{kj} = 1-r_{kj}$. This is illustrated in Fig. 3.1(a). A typical sample function of $R(t_j)$ is shown in Fig. 3.1(b). The moments of this process are


a_{k+1}
a_k
a_{k-1}
r_{kj}
q_{kj}
Δa
t_{j-1}
t_j
t_{j+1}
Δt

(a)


$R(t_j)$
Δa
t
Δt

(b)


a_{k+1}
a_k
a_{k-1}
$q_{k+1,j}$
$r_{k-1,j}$

(c)

Fig. 3.1.

$$\alpha_1(a_k,t_j,\Delta t) = (r_{kj}-q_{kj})\Delta a \tag{44}$$

$$\alpha_2(a_k,t_j,\Delta t) = r_{kj}(\Delta a)^2 + q_{kj}(\Delta a)^2$$

$$= (\Delta a)^2 \tag{45}$$

The limiting random walk process, as $\Delta t \to 0$, can be arranged to have the same infinitesimal moments as $a(t)$ by setting

$$(r_{kj}-q_{kj})\Delta a = A(a_k,t_j)\Delta t \tag{46}$$

$$\Delta a^2 = B(a_k,t_j)\Delta t \tag{47}$$

It is noted that, in the present application, $B(a,t) = B(t)$, independent of a , then from equation (47) (if Δa is fixed in advance) Δt may be computed at every time step, and will, in general, vary as the diffusion process evolves. Since $r_{kj} = 1-q_{kj}$, equation (46) can be used to calculate r_{kj} and q_{kj} at any time t_j . Thus, from equations (46) and (47)

$$r_{kj} = \frac{1}{2}\left[1 + \frac{A(a_k,t_j)}{B(t_j)}\,\Delta a\right] \tag{48}$$

$$q_{kj} = \frac{1}{2}\left[1 - \frac{A(q_k,t_j)}{B(t_j)}\,\Delta a\right] \tag{49}$$

In implementing the random walk scheme one is essentially diffusing probability "mass", over a discrete mesh of space and time. With a deterministic start condition, $a(t) = a_k$ (say), at $t = 0$, one begins with a unit probability mass, located at $(a_k,0)$, and diffuses this mass forward in time, using the scheme illustrated in Fig. 3.1(a). Special care needs to be taken at $a = 0$, since the expressions for r_{kj} and q_{kj} become singular here (Roberts (1978(a))). This difficulty can be overcome by treating $a = 0$ as a reflecting barrier; thus probability mass is reflected off this barrier, ensuring that there is no diffu-

sion of mass into the region $a < 0$. One should also be careful in the choice of Δa and Δt , to ensure that r_{kj} and q_{kj} remain positive, and hence that the random walk process remains physically meaningful.

The random walk scheme can easily be shown to be equivalent to a particular finite difference approximation of the FPK equation. Thus, if $p(k,j)$ is the probability of being in state a_k, t_j , one has (see Fig. 3.1(c))

$$p(k,j+1) = p(k-1,j)r_{k-1,j} + p(k+1,j)q_{k+1,j} \tag{50}$$

This can be rearranged as (using equations (48) and (49))

$$\frac{p(k,j+1)-p(k,j)}{\Delta t}$$

$$= \frac{p(k-1,j)}{2\Delta t}\left[\frac{1+A_{k-1,j}\Delta a}{B_j}\right] + \frac{p(k+1,j)}{2\Delta t}\left[\frac{1-A_{k+1,j}\Delta a}{B_j}\right] - \frac{p(k,j)}{\Delta t} \tag{51}$$

where

$$A_{k,j} = A(a_k,t_j) \quad ; \quad B_j = B(t_j) \tag{52}$$

Rearranging again gives (using equation (47) - i.e. $\Delta a^2 = B_j \Delta t$)

$$\frac{p(k,j+1)-p(k,j)}{\Delta t}$$

$$= -\frac{1}{2\Delta a}\left[p(k+1,j)A_{k+1,j} - p(k-1,j)A_{k-1,j}\right]$$

$$+ \frac{B_j}{2\Delta a^2}\left[p(k+1,j) - 2p(k,j) + p(k-1,j)\right] \tag{53}$$

In the limit, as $\Delta t, \Delta a \to 0$, the left hand side of the above approaches the limit $\partial p/\partial t$. The first term on the right hand side approaches

$$- \frac{\partial}{\partial a}[A(a,t)p] \tag{54}$$

and the last term approaches

$$\frac{1}{2} B(t) \frac{\partial^2 p}{\partial a^2} \tag{55}$$

Since here $B(a,t)$ is independent of a, it follows that equation (53) approaches, in the limit, the FPK equation given by equation (36).

4. Averaging the energy envelope

We now consider an oscillator with light, non-linear damping and a non-linear restoring force. The equation of motion is

$$\ddot{x} + \epsilon^2 h(x,\dot{x}) + g(x) = \epsilon z(t) \tag{56}$$

where, as in Chapter Two, the excitation is scaled with respect to ϵ, to ensure that the response displacement process, $x(t)$, is $O(\epsilon^0)$. As before, we are interested in developing an asymptotic approximation, valid as $\epsilon \to 0$, and accurate when ϵ is small, but finite. However, unlike the analysis based on the standard stochastic averaging method, presented in Chapter Two, it will *not* be assumed here that $g(x)$ is a linear term, plus a non-linearity of order ϵ^2. Instead, the analysis will be based on the assumption that the function $g(x)$ is a *strongly* non-linear term. $z(t)$ will be assumed to be a stationary random process, with zero mean, and a large band-width.

The energy envelope, $E(t)$, may be defined by

$$E(t) = \frac{\dot{x}^2}{2} + V(x) \tag{57}$$

where $V(x)$ is the potential energy function (see also Chapter One and Two). Thus here

$$V(x) = \int_0^x g(\xi)d\xi \tag{58}$$

Following an argument similar to that given in Chapter Two, it can easily be shown from equations (56) and (57), that

$$\dot{E} = P_{in}(t) - P_{dis}(t) \tag{59}$$

where

$$P_{in}(t) = \epsilon z(t)\dot{x} \tag{60}$$

and

$$P_{dis}(t) = \epsilon^2 h(x,\dot{x})\dot{x} \tag{61}$$

These expressions are identical to those given earlier, in Chapter Two, for the case of small non-linearity in $g(x)$, and $P_{in}(t)$ and $P_{dis}(t)$ have the meaning described earlier.

Equation (59) is one of a pair of first-order equations which can be used to describe completely the behaviour of the oscillator, as an alternative to equation (56). For the other member of the pair there are various possibilities, depending on ones choice of a process, $W(t)$ say, to complement $E(t)$. Ideally one would like to find a process, $W(t)$, which is also slowly varying, when ϵ is small, such that the joint vector process, $\underset{\sim}{X}$ is governed by an equation of the form of (2.76). The Stratonovich-Khasminskii theorem would then be applicable, to show that $\underset{\sim}{X}(t)$ converges to a Markov process. Unfortunately, in the case of arbitrary non-linear stiffness, does not seem possible to find a slowly varying process, with a direct physical significance, which can be used to complement $E(t)$.

The difficulty will be overcome here by allowing $W(t)$ to be rapidly varying and by suitably generalising the standard averaging procedure to deal with this departure from the standard form of equation (2.76). Here $W(t)$ is identified as the phase process, $\Phi(t)$ defined by

$$\sqrt{V(x)} = \sqrt{E}\cos\Phi \quad , \quad \dot{x} = -\sqrt{2E}\sin\Phi \tag{62}$$

It is noted that this specification of $E(t)$ and $\Phi(t)$ uniquely defines the position of the oscillator in the phase plane. The appropriate first-order equations for $E(t)$ and $\Phi(t)$ are now

$$\dot{E} = \epsilon^2 h(E,\Phi)\sqrt{2E}\sin\Phi - \epsilon\sqrt{2E}\sin\Phi\, z(t) \tag{63}$$

$$\dot{\Phi} = \frac{\epsilon^2 h(E,\Phi)\cos\Phi}{\sqrt{2E}} - \frac{\epsilon\cos\Phi z(t)}{\sqrt{2E}} + \frac{g(E,\Phi)}{\sqrt{2E}\cos\Phi} \tag{64}$$

where $h(E,\Phi) = h(x,\dot{x})$, $g(E,\Phi) = g(x)$.

In the particular case where the restoring force is *linear*, i.e. $g(x) = \omega_0^2 x$, then

$$\Phi(t) = \omega_0 t + \phi(t) \tag{65}$$

where $\phi(t)$ is *slowly varying*. Equations (63) and (64) then become equivalent to the equations for $a(t)$ and $\phi(t)$, given in Chapter Two (see (2.24) and (2.25)).

4.1 Averaging the energy dissipation term

$P_{dis}(t)$ may be averaged by simply time-averaging over one period of oscillation, on the assumption that $E(t)$ does not change appreciably over such a time interval. The appropriate period to choose here is the period of free oscillation, without

damping, $T(E)$, as given by equation (1.40). The averaged $P_{dis}(t)$, denoted $\bar{P}_{dis}(t)$, is given by

$$\bar{P}_{dis}(t) = -\frac{\epsilon^2}{T(E)} \int_t^{t+T(E)} h(E,\Phi)\ \sqrt{2E}\ \sin\Phi\ dt \qquad (66)$$

and may be written alternatively as

$$\bar{P}_{dis}(t) = -\frac{4\epsilon^2}{T(E)} \int_0^b h[x,\sqrt{2(E-V(x))}]dx = \epsilon^2 B(E) \qquad (67)$$

where

$$V(b) = E \qquad (68)$$

and

$$B(E) = -\frac{4}{T(E)} \int_0^b h[x,\sqrt{2(E-V(x))}]\ dx \qquad (69)$$

Comparison of these expressions with those given in section 4 of Chapter One (see equations (1.39) and (1.40)) shows that $B(E)$ is, apart from the scaling factor ϵ^2 , the loss function, $L(E)$. Thus

$$L(E) = \epsilon^2\ B(E) \qquad (70)$$

4.2 Averaging the energy input term

To average $P_{in}(t)$ it is necessary to generalise, somewhat, the argument given in Chapter Two, in connection with the standard stochastic averaging method (see section 2.3).

It will be assumed here that a time interval, Δt , can be found such that

(1) $\Delta t >> \tau_{cor}$, where τ_{cor} is the correlation time scale of $z(t)$

(2) $E(t)$ and $\Phi(t)$ do not change appreciably from t to $t+\Delta t$.

This last requirement, with respect to Φ , is more restrictive than that imposed earlier, in Chapter Two. It effectively means that τ_{cor} must be appreciably less than the periods of oscillation which occur in realisations of $x(t)$ - i.e.

$$\tau_{cor} << \underset{\text{over } E}{\text{Min}\{T(E)\}} \tag{71}$$

This condition may be compared with that implied in the normal stochastic averaging method, where it is sufficient that τ_{cor} be much less than the correlation time scale of the response - i.e. (see Stratonovich (1964)[1])

$$\tau_{cor} << \frac{1}{\epsilon\omega_0} \tag{72}$$

where, as before, ω_0 is the undamped, linear natural frequency.

To average $P_{in}(t)$, with the above restrictions on $z(t)$, the mean and correlation function of this process will first be evaluated, and then time averaged, using $T(E)$, as before, as the appropriate averaging period.

From equations (60) and (62) one has

$$P_{in}(t) = \epsilon\beta_1(E,\Phi)\ z(t) \tag{73}$$

where

$$\beta_1(E,\Phi) = -\sqrt{2E}\ \sin\Phi \tag{74}$$

Hence

$$P_{in}(t) = \epsilon(\beta_1)_{t-\Delta t} z(t) + \epsilon\left[\left[\frac{\partial \beta_1}{\partial E}\right]_{t-\Delta t} \Delta E + \left[\frac{\partial \beta_1}{\partial \Phi}\right]_{t-\Delta t} \Delta\Phi\right] z(t)$$

$$+ \text{higher order terms in } \Delta E, \Delta\phi \qquad (75)$$

where

$$\Delta E = E(t) - E(t-\Delta t) \qquad (76)$$

$$\Delta\Phi = \Phi(t) - \Phi(t-\Delta t) \qquad (77)$$

and Δt is an interval of time which satisfies the requirements stated above. The expansion of β_1 about its value of $t-\Delta t$ is similar to the expansion employed earlier, in connection with the standard stochastic averaging method (see equation (2.41) to (2.43)).

In the first term on the right hand side of this equation, $z(t)$ is now statistically independent of the value of β_1 at $t-\Delta t$, by virtue of the condition that $\Delta\tau >> \tau_{cor}$; hence the statistics of this term can be readily evaluated. With regard to the second term, one can obtain ΔE and $\Delta\Phi$ by integrating equations (63) and (64), respectively. Hence

$$\Delta E = \epsilon^2 \int_{t-\Delta t}^{t} \alpha_1(E,\Phi)\, d\tau + \epsilon \int_{t-\Delta t}^{t} \beta_1(E,\Phi)\, z(\tau)\, d\tau \qquad (78)$$

and

$$\Delta\Phi = \epsilon^2 \int_{t-\Delta t}^{t} \alpha_2(E,\Phi)\, d\tau + \epsilon \int_{t-\Delta t}^{t} \beta_2(E,\Phi)\, z(\tau)\, d\tau$$

$$+ \int_{t-\Delta t}^{t} \gamma(E,\Phi)\, d\tau \qquad (79)$$

where

$$\alpha_1(E,\Phi) = h(E,\Phi)\sqrt{2E}\,\sin\Phi \tag{80}$$

$$\alpha_2(E,\Phi) = h(E,\Phi)\cos\Phi/\sqrt{2E} \tag{81}$$

$$\beta_2(E,\Phi) = -\cos\Phi/\sqrt{2E} \tag{82}$$

$$\gamma(E,\Phi) = g(E,\Phi)/(\sqrt{2E}\,\cos\Phi) \tag{83}$$

The functions α, β and γ in equations (74), and (80) to (83), can be expanded about their values at $t-\Delta t$. Thus

$$\alpha_1[E(\tau),\Phi(\tau)] = \alpha_1[E(t-\Delta t),\Phi(t-\Delta t)] + \left[\frac{\partial\alpha_1}{\partial E}\right]_{t-\Delta t} \Delta E_\tau + \left[\frac{\partial\alpha_2}{\partial\Phi}\right]_{t-\Delta t} \Delta\Phi_\tau$$

$$\text{higher order terms} \tag{84}$$

and similarly for β_1, α_2, β_2 and γ . $\Delta E_\tau = E(\tau)-E(t-\Delta t)$ and $\Delta\Phi_\tau = \Phi(\tau)-\Phi(t-\Delta t)$ can be found by integrating equations (63) and (64) again. This process can be continued to yield a perturbation series expansion for ΔE and $\Delta\Phi$, in power of ϵ . If one assumes that terms of order Δt are negligible, the resulting expansions are

$$\Delta E = \epsilon(\beta_1)_{t-\Delta t} \int_{t-\Delta t}^{t} z(\tau)\,d\tau + O(\epsilon^2) \tag{85}$$

and

$$\Delta\Phi = \epsilon(\beta_2)_{t-\Delta t} \int_{t-\Delta t}^{t} z(\tau)\,d\tau + O(\epsilon^2) \tag{86}$$

Hence, from equations (75), (85) and (86)

$$P_{in}(t) = \epsilon(\beta_1)_{t-\Delta t} z(t) + \epsilon^2 \left[\frac{\partial \beta_1}{\partial E}\ \beta_1 + \left[\frac{\partial \beta_1}{\partial \Phi}\right]\ \beta_2\right]_{t-\Delta t} \cdot$$

$$\times \int_{t-\Delta t}^{t} z(t) z(\tau) d\tau + 0(\epsilon^3) \qquad (87)$$

The mean of $P_{in}(t)$ can now be evaluated by taking the expectations of both sides of equation (87). Using the fact that

$$\frac{\partial \beta_1}{\partial E}\ \beta_1 + \frac{\partial \beta_1}{\partial \Phi}\ \beta_2 = \left[\frac{-\sin\Phi}{\sqrt{2E}}\right] \left[-\sqrt{2E}\ \sin\Phi\right]$$

$$+ \left[-\sqrt{2E}\ \cos\Phi\right] \left[\frac{-\cos\Phi}{\sqrt{2E}}\right] = 1 \qquad (88)$$

and that $E\{z(t)\} = 0$, one obtains

$$E\{P_{in}(t)\} = \epsilon^2 \int_0^{\Delta t} w_z(u) du + 0(\epsilon^3) \qquad (89)$$

where $w_z(\tau)$ is the correlation function of $z(\tau)$ - i.e.,

$$w_z(\tau) = E\{z(t) z(t+\tau)\} \qquad (90)$$

Since $\Delta\tau >> \tau_{cor}$, the upper limit in the integral in equation (89) can be replaced by ∞ . Then, on neglecting terms of order ϵ^3 or higher (using the basic assumption that ϵ is small) one can write equation (89) as

$$E\{P_{in}(t)\} = \epsilon^2 \pi S_z(0) \qquad (91)$$

where $S_z(\omega)$, the power spectrum of $z(t)$, is given by

$$S_z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w_z(\tau) e^{i\omega\tau} d\tau \qquad (92)$$

In the special case where $z(\tau)$ is an ideal white noise one can write

$$w_z(\tau) = I\delta(\tau) \tag{93}$$

where I is the "strength" of the process. Hence

$$S_z(0) = I/2\pi \tag{94}$$

and equation (91) becomes

$$E\{P_{in}(t)\} = \epsilon^2 I/2 \tag{95}$$

It can be shown (Roberts (1978(b))) that this result is exact if $z(t)$ is a Gaussian white noise. For the linear case this result has previously been derived, in a different way, by Karnopp (1967), who conjectured that it is generally true for non-linear oscillators.

The correlation function of $P_{in}(t)$ is defined by

$$w_p(\tau) = E\{P_{in}(t)P_{in}(t+\tau)\} \tag{96}$$

By using equation (87) again the following expression for $w_p(\tau)$ is obtained:

$$w_p(\tau) = \epsilon^2 E\{(\beta_1)_t^2\}\, w_z(\tau) + 0(\epsilon^3) \tag{97}$$

where the statistical independence of $(\beta_1)_{t-\Delta t}$ and $z(t)$ has been used, together with the fact that β_1 does not change appreciably over an interval Δt . On neglecting terms of order ϵ^3 , and higher order, as in the evaluation of the mean of $P_{in}(t)$, and substituting for β_1 from equation (74), one has the result

$$w_p(\tau) = \epsilon^2 E\{2E(t)\sin^2\Phi(t)\}w_z(\tau) \tag{98}$$

with an error of order ϵ^3 .

Since τ_{cor} is very much smaller than all the other time constants relevant to the oscillator, it is permissable to make the substitution

$$w_z(\tau) = \delta(\tau) \int_{-\infty}^{\infty} w_z(\tau)d\tau = 2\pi\, S_z(0)\,\delta(\tau) \qquad (99)$$

Hence equation (98) may be written as

$$w_p(\tau) = I_p\, \delta(\tau) \qquad (100)$$

where

$$I_p = \epsilon^2\, 2\pi\, S_z(0)\; E\{2E(t)\sin^2\Phi(t)\} \qquad (101)$$

It is now evident that, for sufficiently small values of ϵ and τ_{cor} , $P_{in}(t)$ behaves effectively as a white noise process, with a "strength function" given by equation (101).

The expression for I_p given above can be simplified by using the fact that $E(t)$ does not change appreciably over any one cycle in the response. If the conditional expectation of $2E(t)\sin^2\Phi(t)$, given that $E(t_0) = E$, is denoted

$$E\{2E(t)\sin^2\Phi(t)\,|\,E(t_0) = E\}$$

and the probability density function of $E(t_0)$ is denoted $p(E)$, then one can write

$$E\{2E(t)\sin^2\Phi(t)\} = \int_0^{\infty} E\{2E(t)\sin^2\Phi(t)\,|\,E(t_0) = E\}\; p(E)dE \qquad (102)$$

The conditional expectation can be approximated by averaging over a period of free oscillation, $T(E)$, (as in the case of $P_{dis}(t)$, discussed earlier); thus

$$E\{2E(t)\sin^2\Phi(t)\,|\,E(t_0) = E\}$$

$$\sim \frac{1}{T(E)} \int_{t_0}^{t_0+T(E)} E\{2E(t)\sin^2\Phi(t)\,|\,E(t_0) = E\}dt \qquad (103)$$

Since $E(t)$ changes slowly, one can assume that $E(t) \sim E$, over the integration range in equation (103). Hence

$$E\{2E(t)\sin^2\Phi(t)\,|\,E(t_0) = E\}$$

$$\sim \frac{2E}{T(E)}\, E\left\{ \int_{t_0}^{t_0+T(E)} \sin^2\Phi \; dt \right\} = C(E) \qquad (104)$$

where

$$C(E) = \frac{4\sqrt{2}}{T(E)} \int_0^b \sqrt{E-V(x)}\; dx \qquad (105)$$

This last step is achieved by noting that, from equations (62), $2E\sin^2\Phi = \dot{x}^2$ and that $\dot{x}^2dt = \dot{x}dx = \sqrt{2(E-V(x))}\; dx$. On combining equations (102) and (104) one has the approximate result

$$E\{2E(t)\sin^2\Phi(t)\} = \int_0^\infty C(E)p(E)dE = E\{C[E(t)]\} \qquad (106)$$

and it follows that a good approximation to I_p is

$$I_p = \epsilon^2 2\pi S_z(0)\; E\{C[E(t)]\} \qquad (107)$$

The fact that $P_{in}(t)$ can be represented approximately as a white noise process, with mean value given by equation (91) and a

strength function given by equation (107), suggests that the exact $P_{in}(t)$ process can be approximated by an averaged process of the form

$$P_{in}(t) = \epsilon^2\lambda[E(t)] - \epsilon\mu[E(t)]\xi(t) \tag{108}$$

where $\xi(t)$ is a white noise process of unit strength - i.e.,

$$E\{\xi(t)\xi(t+\tau)\} = \delta(\tau) \tag{109}$$

λ and μ in equation (108) must be chosen such that $P_{in}(t)$ has the correct mean and correlation strength function.

From equation (108), following an analysis similar to that described earlier for $P_{in}(t)$, one obtains

$$E\{P_{in}(t)\} = \epsilon^2 E\{\lambda(E)\} + \frac{\epsilon^2}{2} E\{\frac{d\mu}{dE}\mu\} \tag{110}$$

and

$$W_p(\tau) = \epsilon^2 E\{\mu^2\}\delta(\tau) \tag{111}$$

Hence one must set

$$\mu = [C(E)2\pi S_z(0)]^{\frac{1}{2}} \tag{112}$$

$$\lambda = \pi S_z(0) - \frac{1}{2}\left[\frac{d\mu}{dE}\mu\right] \tag{113}$$

It is noted that the last terms in equations (110) and (113) arise because $\xi(t)$ is being interpreted here as a "physical" process; thus the differential equation for $P_{in}(t)$, given by equation (108) is intepreted in the Stratonovich sense (e.g. see Sobczyk (1985[1])). If equation (108) is written in Itô form - i.e.

$$dP_{in} = \epsilon^2 \lambda dt - \epsilon \mu dW \tag{114}$$

then $w_p(\tau)$ is given by (111), as before, but now

$$E\{P_{in}(t)\} = \epsilon^2 E\{\lambda(E)\} \tag{115}$$

The difference between equations (110) and (115) is the "Wong-Zakai" correction term, and arises because the diffusion coefficient, μ is here dependent on E .

4.3 Complete equation for E(t)

The exact equation for the energy envelope process, $E(t)$, can now be replaced by

$$\dot{E} = \epsilon^2 H(E) - \epsilon J(E) \xi(t) \tag{116}$$

where

$$H(E) = - B(E) + \pi S_z(0)[1 - C'(E)/2] \tag{117}$$

and

$$J(E) = [2\pi S_z(0) C(E)]^{1/2} \tag{118}$$

Here equation (116) is a stochastic differential equation in the Stratonovich sense. In the Itô sense it has the same form, but now

$$H(E) = - B(E) + \pi S_z(0) \tag{119}$$

Equation (116) shows that, when suitably averaged, the energy envelope is a one-dimensional Markov process. Let $p(E|E_0;t)$ be the transition probability density function for $E(t)$. Then using standard methods (see Chapter One), the FPK equation governing $p(E|E_0;t)$ is given by

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial E}\{\epsilon^2 B(E)) - \pi S_f(0)]p\} + \pi S_f(0) \frac{\partial^2}{\partial E^2}[C(E)p] \tag{120}$$

where $S_f(\omega) = \epsilon^2 S_z(\omega)$ is the spectrum of $f(t) = \epsilon z(t)$, the right hand side of equation (56).

4.4 Stationary solution

Since it has been assumed, so far in this discussion, that the excitation is a zero-mean stationary random process, it follows that a stationary solution

$$\lim_{t\to\infty} p(E|E_0;t) = w(E) \tag{121}$$

will exist, where $w(E)$ is the stationary density function for $E(t)$. On setting $\partial p/\partial t = 0$ in equation (120), the following solution for $w(E)$ is obtained

$$w(E) = kT(E)\exp\{-\frac{\epsilon^2}{\pi S_f(0)} Q(E)\} \tag{122}$$

where

$$Q(E) = \int_0^E \frac{B(\xi)}{C(\xi)}\, d\xi \tag{123}$$

and k is a normalisation constant. In the case of linear stiffness, this result reduces to that found earlier, in Chapter Two, using the standard stochastic averaging method (see equation (2.144)).

The relationship between $w(a)$ and $w(x,\dot{x})$, given by equation (2.112), can also be generalised. By a variety of

physical arguments (Stratonovich[1] (1964), Roberts (1977[2], 1981)) it can be shown that

$$w(x,\dot{x}) = \frac{W(E)}{T(E)} \tag{124}$$

Hence, from equations (122) and (124),

$$w(x,\dot{x}) = k \exp\left\{-\frac{\epsilon^2}{\pi S_f(0)} Q\left[\frac{\dot{x}^2}{2} + V(x)\right]\right\} \tag{125}$$

It is interesting to note that, in the case where $h(x,\dot{x})$ is of the form

$$h(x,\dot{x}) = \dot{x}\, f(E) \tag{126}$$

(which includes the case of linear damping) equation (125) agrees with the exact result due to Caughey (see equation (1.83)). In the more general case of arbitrary non-linear damping, equation (125) can be obtained by the method of equivalent non-linearisation, as described in Chapter One (see equations (1.83) and (1.95)). The agreement between equation (125) and the result from equivalent non-linearisation, based on Caughey's exact solution, is due to the fact that, as $\epsilon \to 0$, the exact form of damping becomes unmaterial; all that matters, as far as the distribution of the response is concerned, is the energy loss function, $L(E)$. Thus the averaging method leads to an expression for the distribution of the response in which the contribution from damping is only through the energy loss function. It follows that any exact solution for a particular form of damping will also yield an asymptotically exact solution for any form of non-linear damping. All that is necessary is to match the energy loss function of the particular form to that of the required form. This is, of course, the basis of the equivalent non-linearisation procedure, described in Chapter One.

4.5 Non-stationary solutions

For the special case of linear damping ($\epsilon^2 h(x,\dot{x}) = 2\zeta\omega_0\dot{x}$) and non-linear stiffness, with the power-law form

$$g(x) = k|x|^v \operatorname{sgn}(x) \tag{127}$$

it is possible to obtain an analytical solution for the transition density function $p(E|E_0;t)$ of $E(t)$. For this class of oscillators it is found that

$$\epsilon^2 B(E) = 2\zeta\omega_0 \alpha E \tag{128}$$

$$C(E) = \alpha E \tag{129}$$

where

$$\alpha = \frac{2(v+1)}{(v+3)} \tag{130}$$

Thus the FPK equation becomes (from equation (120))

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial E}\left[[2\zeta\omega_0\alpha E - \pi S_f(0)]p\right] + \pi\alpha S_f(0)\frac{\partial^2}{\partial E^2}(Ep) \tag{131}$$

It is convenient to introduce the non-dimensional energy variable

$$\chi(t) = \frac{E(t)}{\gamma k \sigma^{v+1}} \tag{132}$$

where

$$\gamma = \frac{1}{v+1}\left\{\Gamma\left[\frac{1}{v+1}\right] / \Gamma\left[\frac{3}{v+1}\right]\right\}^{\frac{v+1}{2}} \tag{133}$$

and σ is the stationary standard deviation of $x(t)$. The transition density, $p(\chi|\chi_0;t)$ is found, from equation (131) to be given by (Roberts (1981))

$$p(\chi|\chi_0;t) = \frac{1}{(1-q)}\left[\frac{\chi}{\chi_0 q}\right]^{\frac{\rho}{2}} \exp\left\{-\frac{\chi+\chi_0 q}{1-q}\right\} I_\rho\left\{\frac{2\sqrt{\chi\chi_0 q}}{1-q}\right\}$$

where

$$\rho = \frac{1}{\alpha} - 1 \tag{134}$$

$$q = e^{-2\alpha\zeta\omega_0 t} \tag{135}$$

and $I_\rho(\ )$ is the modified Bessel function of the first kind, of order ρ.

In studying the transient response to suddenly applied stationary random excitation only the marginal probability density function, $p(\chi,t)$, is required. For an initially quiescent oscillator one obtains, from equation (134),

$$p(\chi,t) = \frac{1}{\Gamma(1+\rho)(1-q)^{1+\rho}}\,\chi^\rho \exp\left[-\frac{\chi}{1-q}\right] \tag{136}$$

From this result, by integration, it is easy to find expressions for the moments of $x(t)$. For example, the mean square of the response is given by

$$E\{x^2(t)\} = \sigma_x^2(t) = \sigma^2(1-q)^{\frac{2}{1+v}} \tag{137}$$

For the case of non-stationary excitation a possible representation is

$$f(t) = \epsilon z(t) = a(t)n(t) \tag{138}$$

where $\alpha(t)$ is a deterministic modulating function, and $n(t)$ is a white noise process, with spectral level S_0. One can then replace $S_f(0)$ in the FPK equation for $E(t)$ by $\alpha^2(t)S_0$ (see Roberts and Spanos (1986[1])). The results given by equations (134) and (136) can then be generalised. Specifically, it can be shown that (Spanos (1983))

$$p(E,t|E_1,t_1)$$

$$= \frac{1}{c^*(t_1,t)}\left[\frac{E}{E_1 e^{-2\zeta\alpha\omega_0 t}}\right]^{\frac{\rho}{2}} \exp\left[\frac{-E+E_1 e^{-2\zeta\alpha\omega_0\tau}}{c^*(t_1,t)}\right]$$

$$\times\ I_\rho\left[\frac{2\sqrt{EE_1}\, e^{-\zeta\alpha\omega_0\tau}}{c^*(t_1,t)}\right] \qquad (139)$$

where

$$\tau = t-t_1 \qquad (140)$$

and

$$c^*(t_1,t) = \alpha_\pi S_0 e^{-2\alpha\zeta\omega_0 t}\int_{t_1}^{t_2} \exp(2\zeta\alpha\omega_0 z)\alpha^2(z)dz \qquad (141)$$

For more general situations, where $g(x)$ is not of power-law form, a numerical solution of the FPK equation for $E(t)$ is necessary. A random-walk based numerical scheme has been outlined (Roberts (1981)) and specific results have been obtained for the transient response of a Duffing oscillator to suddenly applied white noise. Very good agreement with digital simulation estimates of the variation of the mean square response with time was obtained. An alternative, explicit finite difference scheme has also been described (Roberts (1986(a))).

4.6 Parametric excitation

The energy envelope averaging method can also be applied to oscillators with parametric excitation. Zhu (1983(a),(b)) has analysed the case of an oscillator with the following equation of motion:

$$\ddot{x} + \epsilon^2 h(x,\dot{x}) + g(x) = \epsilon \sum_{i=1}^{n} h_i(x,\dot{x})\xi_i(t) \qquad (142)$$

Here ξ_i $(i = 1,2,\ldots,n)$ are taken to be mutually independent stationary processes, with zero mean and broad-band character. An FPK equation was derived which represents a generalisation of equation (120). The steady-state probability density functions for $E(t)$ were evaluated for a number of specific cases, and appropriate stability criteria were derived.

A special case of equation (142) was analysed earlier by Dimentberg (1980), who compared his theoretical results with digital simulation estimates.

4.7 Non-white excitation

In principle the energy envelope, $E(t)$, should converge to a Markov process, irrespective of the shape of the input spectrum although, of course, the nature of the input will affect the rate of convergence. This suggests that, for oscillators with light damping, it should be possible to derive an FPK equation for $E(t)$ which takes into account the shape of the input spectrum.

Roberts (1983) has described an intuitive approach which does indeed lead, after certain approximations, to the following FPK equation for the energy envelope of the non-linear oscillator governed by equation (56),

$$\frac{\partial p}{\partial t} = -\epsilon^2 \frac{\partial}{\partial E}[m(E)p] + \frac{\epsilon^2}{2}\frac{\partial^2}{\partial E^2}[D(E)p] \qquad (143)$$

Here the drift coefficient, $m(E)$, and the diffusion coefficient, $D(E)$, are given by

$$m(E) = -B(E) + \frac{\pi}{2} \sum_{n=1}^{\infty} (s_n^2 + c_n^2)\, S_z[n\omega(E)] \tag{144}$$

$$D(E) = 2\pi E \sum_{n=1}^{\infty} s_n^2 S_z[n\omega(E)] \tag{145}$$

Here $B(E)$ is given by equation (69) and $S_z(\omega)$ is the power spectrum of $z(t)$. The Fourier coefficients s_n, c_n $(n = 1,2,\ldots)$ are related to the variation of the phase with time, in the case of undamped, free vibration. $\omega(E)$ is the frequency of undamped free vibrations, at energy level E.

It can be shown that, in the special case of stationary white noise excitation, equation (143) is identical to equation (120). Moreover, for oscillators with linear stiffness, and non-white inputs, equation (143) is equivalent to the FPK equation for $a(t)$, obtained by the standard stochastic averaging method (see equation (2.103)).

In many applications the first terms in the summations in equations (144) and (145) are dominant: One can then, to a very good approximation, evaluate $m(E)$ and $D(E)$ as follows:

$$m(E) = -B(E) + \frac{\pi}{2} (s_1^2 + c_1^2)\, S[\omega(E)] \tag{146}$$

$$D(E) = 2\pi E s_1^2 S[\omega(E)] \tag{147}$$

Some comparisons with digital simulation results for the case of a Duffing oscillator with non-white stationary excitations indicate that equation (143) leads to a good approximation for the stationary density function of the energy envelope.

Moreover, the influence of the shape of the input spectrum on the response distribution is well predicted.

Recently a modification of this theoretical approach has been proposed which takes further account of the shape of the input spectrum (Roberts (1984,1986(b))). To appreciate the limitation of the original theory it is sufficient to consider the special case of a linear restoring moment. Then $g(x) = \omega_0^2 x$, $s_1 = c_1 = 1$ and $s_n = c_n = 0$ for $n > 1$. Also $\omega(E) = \omega_0$. Hence equations (144) and (145) reduce to

$$m(E) = - B(E) + \pi S_z(\omega_0) \tag{148}$$

$$D(E) = 2\pi E S_z(\omega_0) \tag{149}$$

Thus, here only the value of the input spectrum at the natural frequency of oscillation is required; i.e., the basic assumptions of the theory are equivalent to making a white noise approximation for $z(t)$.

For the linear case the error in approximating the input as a white noise process can be readily calculated from the standard theory of linear systems. Thus the mean square response is given exactly by

$$\sigma^2 = 2\pi \int_{-\infty}^{\infty} |\alpha(\omega)|^2 S_f(\omega) d\omega \tag{150}$$

where $\alpha(\omega)$ is the frequency response function of the system, given by

$$\alpha(\omega) = \frac{1}{\omega^2 - \omega_0^2 + 2i\zeta\omega\omega_0} \tag{151}$$

where ζ is the usual critical damping factor. σ_w^2 , the white noise approximation for σ^2 , is given by

$$\sigma_w^2 = 2\pi S_f(\omega_0) \int_{-\infty}^{\infty} |\alpha(\omega)|^2 d\omega$$

$$= \frac{\pi S_f(\omega_0)}{2\zeta\omega_0^3} \qquad (152)$$

A value r can be defined as the ratio of the exact to the appropriate result. Thus

$$r = \sigma^2/\sigma_w^2 \qquad (153)$$

The ratio r can be used to correct, in an appropriate fashion, the theory outlined above, for the general case of non-linear stiffness and damping. The idea is to replace $S_z(\omega_0)$ by a modified spectrum

$$S_z'(\omega_0) = rS_z(\omega_0) \qquad (154)$$

Of course, in the linear case, this will have the effect that σ^2 is now given exactly. In the non-linear case one must use an equivalent linear system.

An appropriate form of equivalent linear system is

$$\ddot{x} + 2\zeta(E)\omega(E)\dot{x} + \omega^2(E)x = \epsilon z(t) \qquad (155)$$

where $\omega(E)$ is, as before, the natural frequency at energy level E and $\zeta(E)$ is an amplitude dependent damping factor. The ratio r, computed on this basis will, of course, depend on E.

A suitable method of finding $\zeta(E)$ is to equate it to the damping function $Q(E)$ defined in Chapter One (see equation 1.34). With this definition, $\zeta(E)$ is directly connected to the function $B(E)$, (see equations (1.34) and (70)).

The proposed modification to the Markov theory is a rough method of accounting to some extent, for the effect of the shape of the input spectrum on the distribution of the response. Comparisons between theoretical results and simulation estimates, for an oscillator with linear-plus-quadratic damping, and a softening spring, show that the modified theory leads to a considerable improvement in accuracy.

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AVERAGING METHODS IN RANDOM VIBRATION

J. B. Roberts

CHAPTER FOUR

Application of the Stochastic Averaging Method to the First-Passage Problem

1. Introduction

In practical applications involving the random vibration of a mechanical or structural system it is often required to estimate the probability that the systems's response will stay within safe, prescribed limits, within a specified interval of time. The determination of such a probability is usually called the "first passage problem", and has been extensively studied, particularly during the last two decades (see Roberts (1986(a))).

In this Chapter the existing state of knowledge regarding the first-passage problem, which has been obtained by modelling the response as a diffusion process, will be reviewed. Due to analytical and numerical limitations and difficulties, almost the entire existing body of knowledge is confined to the case of single degree of freedom systems, or oscillators, driven by wide-band excitation. For this class of system the diffusion approach does, however, enable a number of aspects to be treated which are difficult, if not impossible, to tackle by other methods: these include the effects of

(a) non-linearities in both damping and stiffness,
(b) parametric excitation,
(c) non-stationary excitation.

To provide a background to the role of the stochastic averaging method, as a mean of attacking the first-passage problem, available knowledge concerning the general behaviour of first-passage statistics will first be discussed. This will be followed by a detailed examination of the difficulties involved in finding exact solutions, in the case of an oscillator driven by stationary white noise. It will then be shown that many of these difficulties can be overcome by applying the stochastic averaging technique. This approach has the principal effect of reducing the relevant diffusion process from two dimensions to one dimension, with a consequent vast reduction in the analytical difficulties associated with computing first-passage statistics.

2. First-passage statistics

Normally, in practical applications, there is a "safe region" of operation, the outer limits of which are defined by a suitable "barrier". The first-passage problem may be stated thus: To find the probability, $P(t)$, that a response process, $x(t)$, of a system crosses some critical barrier (i.e. exits from the safe domain) at least once in some interval of time 0-t .

It is noted that $P(t)$ will depend on the initial condition of the system, at $t = 0$. Associated with $P(t)$ is the "reliability function"

$$Q(t) = 1 - P(t) \tag{1}$$

$Q(t)$ is the probability that $x(t)$ stays within the safe domain, in the interval 0-t .

In connection with $P(t)$ it is useful to define the first-passage density function $p(t)$, where $p(t)dt$ is the probability that $x(t)$ *first* exceeds the barrier between time t and $t+dt$. It is easy to show that

$$P(t) = \int_0^t p(\xi)d\xi \tag{2}$$

$p(t)$ is simply the density function of the random time, T, to first-passage failure. The moments of T may be expressed as

$$M_n = E(T^n) = \int_0^\infty t^n p(t)dt \tag{3}$$

Of these moments the first, M_1, which is simply the mean time to failure, is by far the most important (for reasons which will appear shortly).

Although complete, exact analytical solutions for $P(t)$ are generally unavailable, its asymptotic behaviour in the case of stationary excitation is well understood. For stationary excitation the response, $x(t)$, will approach a stationary distribution, as the elapsed time becomes large, irrespective of initial conditions, if the system is stable. Under these conditions one can show (e.g. see Roberts (1974)) that

$$Q(t) \rightarrow \exp(-\alpha t) \qquad \text{as} \quad t \rightarrow \infty \tag{4}$$

where α is called the "limiting decay rate". When M_1 is very large, equation (4) is a good approximation for nearly all values of t and it follows, from equations (1) to (4), that

$$\alpha \rightarrow \frac{1}{M_1} \qquad \text{as} \quad M_1 \rightarrow \infty \tag{5}$$

Thus, for very long mean times to failure, the single statistic, M_1, can be used to form an approximation to $P(t)$.

In this connection it is noted that Cramer (1966) and others

have shown that for a Gaussian process the distribution of barrier crossings is asymptotically Poisson distributed as the critical level (b) becomes very large; i.e.,

$$Q(t) \rightarrow \exp(-\nu t) \qquad \text{as} \quad b \rightarrow \infty \tag{6}$$

where ν is the average number of barrier crossings (from within the safe region, to outside the region) per unit of time. This implies that

$$\alpha \rightarrow \nu \qquad \text{as} \quad b \rightarrow \infty \tag{7}$$

There is some evidence to suggest that equation (7) may be valid for non-Gaussian processes (Roberts (1986(a))).

3. Specific first-passage problems

Two types of barrier are of primary importance - single-sided and double-sided barriers. The first of these is such that any value of $x(t)$ less than a fixed level, b say, is safe. Thus first-passage failure occurs when $x(t)$ *first* exceeds b . The second type is such that, for safe operation $|x(t)| < b$.

For oscillators, domains of safe operation are conveniently depicted in the phase-plane. Single and double-sided barriers then appear as shown in Figs. 4.1(a) and (b), where the phase plane coordinates are a scaled velocity $\dot{x}/\omega_0$ (where ω_0 is the natural frequency of small, linearized oscillations) and x . A third type of barrier - the "circular barrier" - is of considerable theoretical interest. Here the safe domain is within a circle in the phase-plane, of radius b , where

$$b^2 = x^2 + \dot{x}^2/\omega_0^2 \tag{8}$$

This barrier is shown in Fig. 4.1(c).

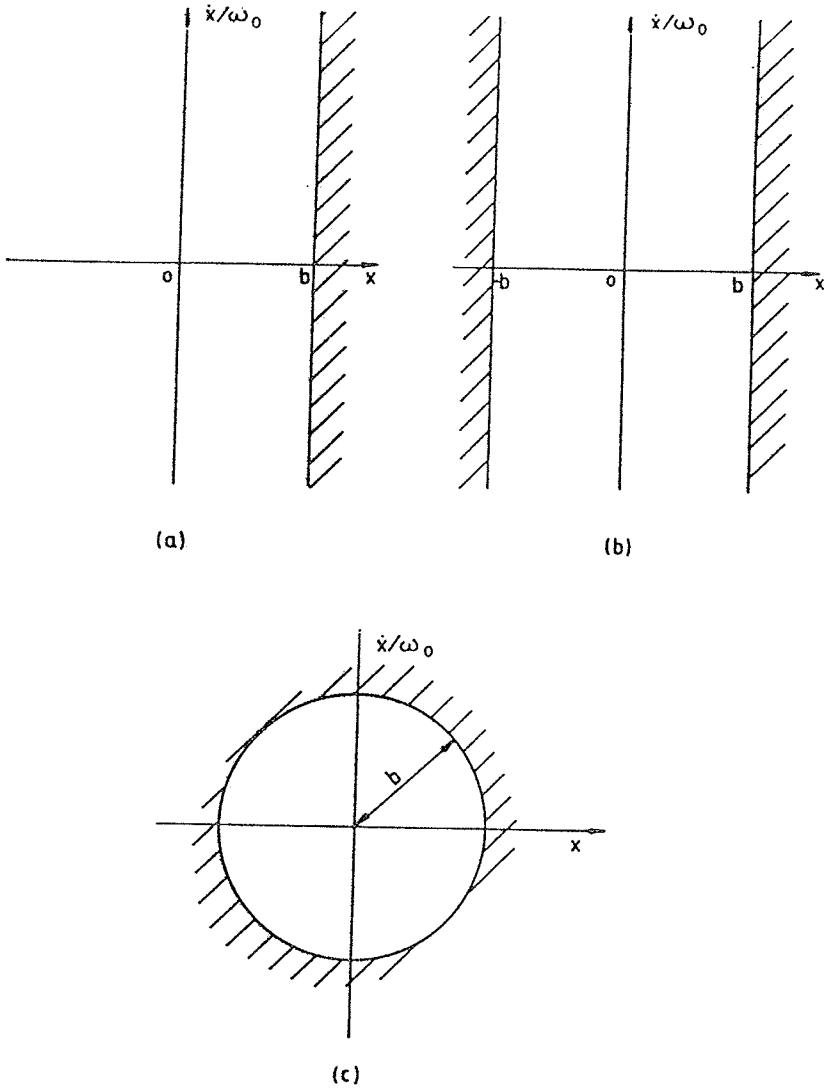


Fig. 4.1.

4. The "exact" approach

We now consider an oscillator with the following equation of motion:

$$\ddot{x} + g(x,y) = f(t) \tag{9}$$

Here $x(t)$ is the displacement response, $y = \dot{x}$, $g(x,y)$ is a general, non-hereditary function of displacement and velocity, and $f(t)$ will be assumed to be a zero-mean, stationary white noise process.

4.1 Diffusion equations

As discussed in Chapter One, the joint process $[x(t),y(t)]$ is a vector, two-dimensional continuous Markov process, with a transition probability density function, $p(x,y|x_0,y_0;t)$ governed by an FPK equation (see equation (1.77)). In the context of first-passage problems it is more convenient to consider the conditional transition density function $q(x,y|x_0,y_0;t)$, where $q(x,y|x_0,y_0;t)dxdy$ is the probability that $x \le x(t) \le x+dx$, $y \le y(t) \le y+dy$, at time t, without departing from the safe domain. Both conditional and unconditional transition density functions are governed by the *same* FPK equation. Thus, from equation (1.77)

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial y}[g(x,y)q] - y\frac{\partial q}{\partial x} + \frac{I}{2}\frac{\partial^2 q}{\partial y^2} \tag{10}$$

where I is the "strength" of the white noise process - i.e.,

$$E\{f(t)f(t+q)\} = I\delta(\tau) \tag{11}$$

The FPK equation can be considered as a continuity equation for the "flow" of "probability mass". This is evident if equation (10) is re-written as

$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}(F_x) + \frac{\partial}{\partial y}(F_y) = 0 \tag{12}$$

where

$$F_x = yq \tag{13}$$

$$F_y = -g(x,y)q - \frac{I}{2}\frac{\partial q}{\partial y} \tag{14}$$

F_x is the flow of probability mass in the x-direction, and F_y is the corresponding flow in the y-direction.

To solve equation (10) one requires a suitable initial condition. Normally a deterministic condition is appropriate - i.e. one knows that $x(t) = x_0$, $y(t) = y_0$, at $t = 0$. Thus

$$\lim_{t\to 0} q(x,y|x_0,y_0;t) = \delta(x-x_0)\delta(y-y_0) \qquad (15)$$

A "stationary start" condition is also sometimes appropriate; here it is assumed that there is an ensemble of initial values, x_0 , y_0 , distributed according to the stationary response distribution.

Solving equation (10) enables one to determine how the probability mass "diffuses" with time in the phase plane, from some initial condition. The diffusion process is, alternatively, governed by an integral equation (the Chapman-Kolmogorov-Smoluchowski (CKS) equation); this may be written as follows:

$$q(x,y|x_0,y_0;t) = \int\int_R q(x,y|x',y';t-t')q(x',y'|x_0,y_0;t')dx'dy' \qquad (16)$$

where the integration range, R , is the safe domain in the phase plane. Equations (10) and (16) are equivalent mathematical representations of the same underlying diffusion law for the process $[x(t),y(t)]$.

Associated with the FPK equation is the adjoint form, usually referred to as the backward Kolmogorov equation. This is of the form

$$\frac{\partial q}{\partial t} = \mathscr{L} q \qquad (17)$$

where $\mathscr{L}$, the "backward operator", is here given by

$$\mathscr{L} = -g(x_0,y_0)\frac{\partial}{\partial y_0} + y_0\frac{\partial}{\partial x_0} + \frac{I}{2}\frac{\partial^2}{\partial y_0^2} \qquad (18)$$

This equation can be used to derive a differential equation for the reliability function $Q(t|x_0,y_0)$ - this is the probability that first-passage will *not* occur in the interval $0-t$, for trajectories in the phase-plane starting at x_0,y_0. Clearly,

$$Q(t|x_0,y_0) = \int_R\int q(x,y|x_0,y_0;t)\,dxdy \tag{19}$$

Thus, by integrating equation (17) with respect to x and y, over R, one obtains

$$\frac{\partial Q}{\partial t} = \mathscr{L}Q \tag{20}$$

From equation (1) it follows that $P(t|x_0,y_0)$ is governed by the same differential equation as $Q(t|x_0,y_0)$.

From equation (20) it is fairly easy to derive the following set of differential equations (usually called the generalised Pontriagin-Vitt (GPV) equations)

$$\mathscr{L}M_n = -nM_{n-1} \qquad (n = 0,1,2\ldots) \tag{21}$$

Since

$$M_0 = \int_0^\infty p(t)dt = 1 \tag{22}$$

equations (21) can be solved successively, to yield M_1, M_2, etc. Of principal interest is the Pontriagin-Vitt (PV) equation for the mean time to failure, M_1. Thus, setting $n = 1$ in equation (21), and using equation (22), one has

$$\mathscr{L}M_1 = -1 \tag{23}$$

4.2 Boundary conditions

To solve the FPK equation for q (see equation (10)) it is necessary to specify boundary conditions. These conditions play a vital role in determining first-passage statistics.

For a single-sided barrier, it is observed that the flow of probability mass in the x direction, F_x (see equation (13)) is such that flow from the safe, to the unsafe region, at $x = b$, occurs only if $y > 0$. For $x = b$ and $y < 0$, F_x is negative, indicating a "return flow", from the unsafe to the safe region. To ensure that the return flow does not occur it is sufficient to set

$$q = 0 \qquad \text{for } x = b,\ y < 0 \tag{24}$$

Similar arguments can be used to specify boundary conditions for double-sided and circular barriers. It can be shown that these conditions lead to well-posed problems (Yang and Shinozuka (1970)).

An alternative approach is to consider solving the differential equation governing Q, as given by equation (20). For the single-sided barrier, for example, plausible boundary conditions for Q are as follows:

$$\left.\begin{aligned} Q(0|x_0,y_0) &= 1 && (x_0,y_0) \in R \\ Q(t|b,y_0) &= 0 && y_0 > 0 \\ Q(t|x_0,y_0) &\to 0 && |y_0| \to \infty \end{aligned}\right\} \tag{25}$$

The first condition expresses the fact that, within a zero time interval, the probability of not exiting from the safe domain is unity (for starts within the safe domain). The second condition corresponds to the fact that if the diffusion process starts on the line $x = b$, $y > 0$, the oscillator response will immediately move out of the safe domain, and thus the probability of not exceeding the barrier is zero. The third condition implies that, if the velocity is infinite, first-passage failure is bound to

occur, within any given time interval. It can be shown that these boundary conditions lead to a well-posed problem for the determination of Q. Similar boundary conditions can be postulated for double-sided and circular barriers (see Roberts (1986(a))).

4.3 Exact analytical solutions

Exact solutions have so far been obtained for only very special, reduced cases where one or more terms in the equation of motion are neglected.

For two-dimensional diffusion processes two specific analytical solutions have been obtained which have some relevance to the general problem. Franklin and Rodemich (1968) have considered the mean time to failure of a randomly accelerated free particle, in the case of a double-sided barrier. The equation of motion here reduces to

$$\ddot{x} = f(t) \tag{26}$$

(i.e. $g(x,y) = 0$ in equation (9). Thus the Pontriagin-Vitt for M_1 (see equations (18) and (23)) is

$$\frac{I}{2}\frac{\partial^2 M_1}{\partial y_0^2} + y_0 \frac{\partial M_1}{\partial x_0} = -1 \tag{27}$$

A complete analytical solution to this problem was obtained by Franklin and Rodemich, for the case where $I = 1$ and $b = 1$. Some idea of the complexity of this solution can be gained by examining the result for $y_0 = 0$; this is as follows:

$$M_1(x_0,0) = C(1-x_0^2)^{\frac{1}{6}} F\left\{-\frac{1}{3},1,\frac{7}{6},\frac{1}{2}(1-x_0)\right\} + F\left\{-\frac{1}{3},1,\frac{7}{6},\frac{1}{2}(1-x_0)\right\} \tag{28}$$

where

$$C = 3^{11/6}/[2\Gamma(\tfrac{1}{3})] \tag{29}$$

and F is the hypergeometric function. For a general barrier level b, the solution for M_1, denoted by $M_1(x_0,y_0,b)$, is related to that for $b = 1$, $M_1(x_0,y_0)$, as follows:

$$M_1(x_0,y_0,b) = b^{2/3}\, M_1(b^{-1}x_0, b^{-\frac{1}{3}}y_0) \tag{30}$$

In view of the complexity of this solution it is, perhaps, not surprising that exact analytical solutions are not available for the more general case of non-zero $g(x,y)$, even for just the mean time to failure, M_1.

In a quite different approach, Kozin (1983) has pointed out that a result due to Dynkin (1965) may be used to obtain specific exact results, in some circumstances. If $h(x,y)$ is some function of x and y then Dynkin's formula may be written as

$$E\{h(x,y)_t\} - E\{h(x,y)_{t_0}\}$$

$$= E\left\{\int_{t_0}^{t} \mathscr{L}h[x(s),y(s)]\, ds\right\} \tag{31}$$

where $\mathscr{L}$ is the backward operator and t is any random time; thus t can be the first-passage time T. If $\mathscr{L}[h(x,y)]$ is just a constant, C say, then,

$$E\left\{\int_{t_0}^{T} \mathscr{L}h[x(s),y(s)]\, ds\right\} = C[E\{T\} - E\{t_0\}] \tag{32}$$

and it follows that equation (31) gives an expression for the mean time to first-passage failure, $M_1 = E\{T\}$.

As an illustration, consider the case of an undamped oscillator, with an equation of motion

$$\ddot{x} + g(x) = f(t) \tag{33}$$

Let

$$h(x,y) = \frac{y^2}{2} + V(x) = E(t) \tag{34}$$

where

$$V(x) = \int_0^x g(\xi)d\xi \tag{35}$$

Clearly, as indicated $h(x,y)$ is here just the total energy of the oscillator. From equations (18) and (34) one has

$$\mathscr{L}[h(x,y)] = \frac{I}{2} \tag{36}$$

and hence equation (31) will give an expression for M_1. For a deterministic initial start condition, i.e., $x(0)$, $y(0)$ and $E(0)$ are known at $t_0 = 0$, then the result is

$$E\{E(T) - E(0)\} = \frac{I}{2} M_1 \tag{37}$$

If T is the time to reach a constant energy level, E, then $E\{E(T)\} = E$ and equation (37) gives

$$M_1 = \frac{2}{I} [E - E(0)] \tag{38}$$

It is interesting to note that, in this simple and exact result, the non-linearity function $g(x)$ does not appear explicitly.

If damping is introduced into the equation of motion then Dynkin's formula does not lead to simple results. However, it may

be possible to use this formula as a method of generating approximate results.

Although the exact results described in this section are not, of themselves, of much practical use in random vibration studies, they are of great value in an indirect way, as a means of partially validating both numerical methods of solving the diffusion equations and simulation methods for estimating first-passage statistics.

4.4 Approximate analytical solutions

If the domain of safe operation is a closed region then it is possible to solve the PV equation for M_1 by a Galerkin technique, as Bolotin (1965) has pointed out. Results have been obtained, using this approach, for the mean time to snap-through of thin curved panels responding to random excitation. The method may be extended to a consideration of the GPV equations for M_n $(n = 1,2,...)$, enabling M_1, M_2, etc. to be computed recursively.

4.5 Numerical solutions

The first numerical solutions of the first passage problem, for the case of a simple linear oscillator, were obtained by Crandall et al, (1966), who solved the CKS equation (see equation (10)) numerically; their methods amounts to diffusing probability mass over a discretised phase-plane, using discrete time steps; the unconditional transition density function, $p(x,y|x_0,y_0;t)$, was used to redistribute the probability mass at each time step, and probability mass which diffused outside the safe domain was counted as "lost".

For non-linear oscillators this approach can not be used, as the transition density function $p(x,y|x_0,y_0;t)$ is generally not known. However, a discrete random analogue can be constructed,

which can be viewed as a finite difference approximation of the governing FPK equation for $q(x,y|x_0,y_0;t)$. Toland and Yang (1971) were able to obtain results by this technique for various types of non-linear oscillators.

A natural alternative to finite difference schemes is the use of finite elements. In a recent series of publications, Bergman and co-workers have obtained numerical solutions to both equations (20) and (21), by using the Petrov-Galerkin finite element method (Bergman and Heinrich (1980,1981,1982), Bergman and Spencer (1983)). A feature of the finite element numerical method, as with other numerical methods, is that the computational effort increases quite sharply as the barrier level increases. In fact the number of finite elements increases as the barrier level squared, and the cost of computation increases as the fourth power of the barrier level.

5. Stochastic averaging approximations

For an oscillator with light non-linear damping, and linear stiffness, we have seen that an appropriate equation of motion is as follows

$$\ddot{x} + \epsilon^2 h(x,y) + \omega_0^2 x = \epsilon z(t) = f(t) \tag{39}$$

and that the amplitude process, $a(t)$, where

$$a^2(t) = x^2 + \frac{y^2}{\omega_0^2} \tag{40}$$

converges to a Markov process, as $\epsilon \to 0$. The limiting stochastic equation for $a(t)$ is, from an application of the Stratonovich-Khasminskii limit theorem (see also equation (2.66))

$$\dot{a} = -\frac{\epsilon^2}{\omega_0} F(a) + \frac{\pi S_f(\omega_0)}{2a\omega_0^2} + \frac{(\pi S_f(\omega_0))^{\frac{1}{2}}}{\omega_0} \xi(t) \qquad (41)$$

where ξ is a white noise process with unit strength and

$$F(a) = -\frac{1}{2\pi}\int_0^{2\pi} h(a\cos\Phi, -a\omega_0\sin\Phi)\sin\Phi d\Phi \qquad (42)$$

(see equations (2.26) and (2.29)).

If $q(a|a_0;t)$ is the *conditional* transition density function for $a(t)$, then it follows from equation (41) that this function is governed by the following FPK equation (which is the same equation that governs the unconditional transition density function, $p(a|a_0;t)$ - see equation (2.103)):

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial a}\left[\left\{ \frac{\epsilon^2 F(a)}{\omega_0} - \frac{\pi S_f(\omega_0)}{2a\omega_0^2} \right\} q \right] + \frac{\pi S_f(\omega_0)}{2\omega_0^2} \frac{\partial^2 q}{\partial a^2} \qquad (43)$$

The associated backward equation is of the general form of equation (17), where here

$$\mathscr{L} = -\left[\frac{\epsilon^2 F(a)}{\omega_0} - \frac{\pi S_f(\omega_0)}{2a_0\omega_0^2} \right] \frac{\partial}{\partial a_0} + \frac{\pi S_f(\omega_0)}{2\omega_0^2} \frac{\partial^2}{\partial a_0^2} \qquad (44)$$

From this, differential equations for the reliability function Q, and the moments, M_n, of T can be derived. They are of the general form of equations (20) and (21), where now the one-dimensional operator, $\mathscr{L}$, is given by equation (44). It is noted that, in this case, the GPV equations for M_n are *ordinary* differential equations, and are thus much easier to solve than their two-dimensional counterparts.

In the particular case of linear damping $\epsilon^2 h(x,y) = 2\zeta\omega_0 y$ and hence

$$\epsilon^2 F(a) = \zeta\omega_0^2 a \tag{45}$$

(as noted earlier, in Chapter 2 - see equation (2.35)). For the linear oscillator it is convenient to introduce a non-dimensional amplitude

$$A = a/\sigma \tag{46}$$

where σ is the standard deviation of the stationary response, $x(t)$. As shown in Chapter Two, the stochastic averaging method leads to the usual white noise approximation for σ, as given by equation (2.117). If t is also replaced by the non-dimensional time

$$\tau = \omega_0 t \tag{47}$$

then, for the linear oscillator, equations (41), (43) and (44) may be written in the following non-dimensional form:

$$\dot{A} = -\zeta\left[A - \frac{1}{A}\right] + \sqrt{2\zeta}\,\xi(t) \tag{48}$$

$$\frac{1}{\zeta}\frac{\partial q}{\partial \tau} = -\frac{\partial}{\partial A}\left[\left[A - \frac{1}{A}\right]q\right] + \frac{\partial^2 q}{\partial A^2} \tag{49}$$

$$\mathscr{L} = \zeta\left[-\left[A - \frac{1}{A}\right]\frac{\partial}{\partial A} + \frac{\partial^2}{\partial A^2}\right] \tag{50}$$

5.1 Boundary conditions

Of the three types of barrier considered earlier, it is clear that the circular barrier is the easiest to deal with, when working with the amplitude process, $a(t)$. Thus the probability that the response stays within the circular barrier is simply the probability that $a(t)$ stays below the level b, where $x^2+y^2/\omega_0^2 = b^2$. For the FPK equation the appropriate boundary condition for $q(a|a_0;t)$ is given by

$$q = 0 \qquad \text{for} \quad a = b \tag{51}$$

whereas, for the reliability function $\phi(t|a_0)$ one must have

$$\left.\begin{aligned} \phi(0|a_0) &= 1 \qquad a_0 < b \\ \phi(t|b) &= 0 \qquad t < 0 \end{aligned}\right\} \tag{52}$$

The boundary conditions for $M_n(a_0)$ are

$$M_n(b) = 0 \tag{53}$$

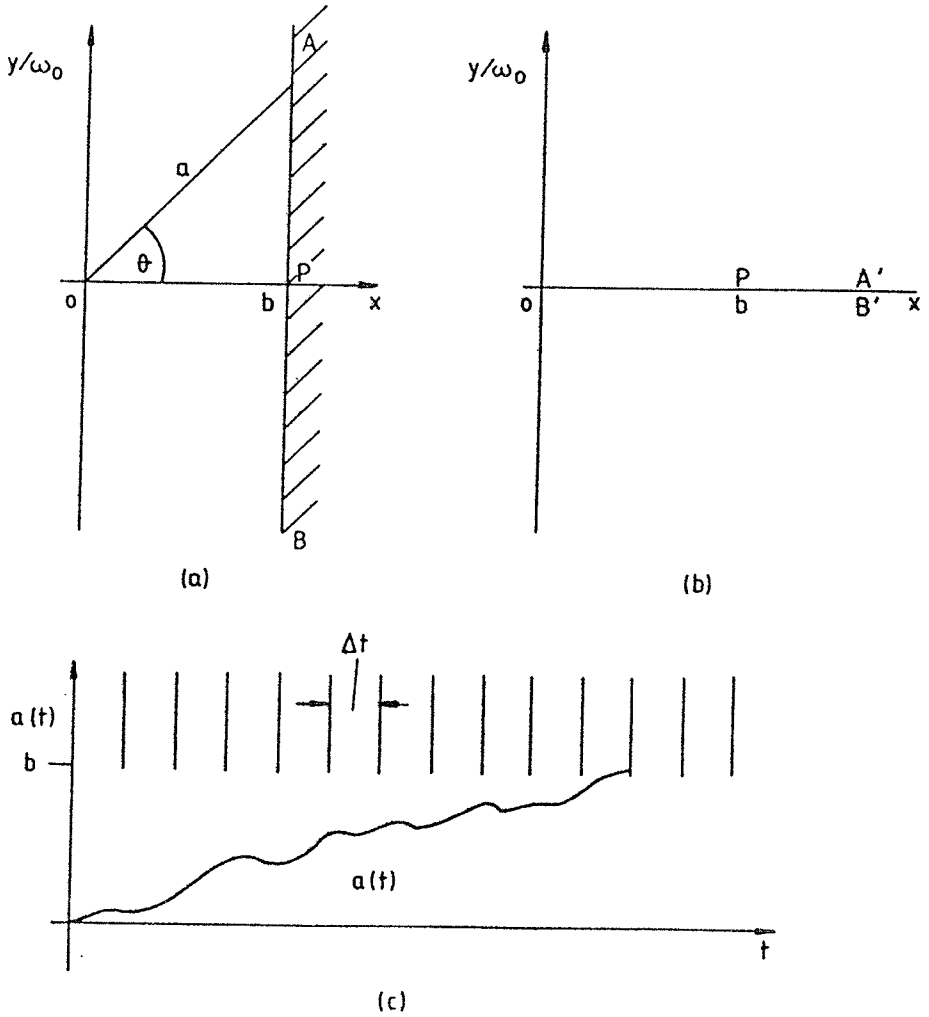


Fig. 4.2.

In addition it is necessary to ensure that $\phi(t|0)$ and $M_n(0)$ are finite.

In the case of the single-sided barrier, failure will occur when $a(t) = b \sec\phi(t)$ (see Fig. 4.2(a)). This rather complicated condition can be simplified by replacing the original barrier by the one shown in Fig. 4.2(b). Here the segments PA, PB of the original barriers are folded back, to lie along the horizontal axis. For light damping, response trajectories will be roughly circular and the probability that a trajectory reaches the barrier in Fig. 4.2(b), without crossing the barrier in Fig. 4.2(a), and vice-verse, is negligible. Thus, first-passage times for the barriers shown in Figs. 4.2(a) and (b) should be almost identical if the damping is light.

For the modified single-sided barrier failure occurs only when $\theta(t) = 0$, and the times at which this occurs will have a small dispersion about the equi-spaced times $t_n = 2_n\pi/\omega_0$. Thus, a reasonable approximation is to adopt the barrier shown in Fig. 4.2(c), for the $a(t)$ process. Here failure occurs only at the discrete times, t_n. This is equivalent to replacing the continuous time envelope (CE process) by a discrete time envelope (DE process), for the purpose of first-passage analysis. For the DE process one can write

$$q(a_n|a_0;t_{n+1}) = \int_0^b p(a|a';\Delta t)q(a'|a_0;t_n)da' \qquad (54)$$

(compare with equation (16), for the two-dimensional case), where $\Delta t = 2\pi/\omega_0$. If $p(a|a';\Delta t)$ is known, equation (54) can be marched in time, from some prescribed initial condition, to determine the evolution of $q(a|a_0;t)$. Hence the survival probability

$$Q(t|a_0) = \int_0^b q(a|a_0;t)da \qquad (55)$$

can be computed, at every time step. It is noted that the vertical barrier lines in Fig. 4.2(c) act as "absorbers" of probability mass, which block further diffusion.

For a double-sided barrier, very similar approximations can be made and one is led to equation (54) again, where now $\Delta t = \pi/\omega_0$ - i.e. one half of the time interval appropriate for a single-sided barrier.

From the nature of these approximations for single and double-sided barriers, and the fact that $p(a|a_0;t)$ depends on the product $\epsilon^2 t$, two important conclusions can be drawn regarding the behaviour of the resulting estimates of $Q(t|a_0)$ (and hence $P(t(a_0))$).

(1) $Q(t|a_0)$ will depend only on b, $\mu\epsilon^2$ and $\epsilon^2 t$, where $\mu = 1$ for single-sided barriers and $\mu = 1/2$ for double-sided barriers.

(2) $Q(t|a_0)$ values for the DE process approach the corresponding values for the CE process as $\epsilon^2 \to 0$.

5.2 Exact analytical solutions

From an analytical viewpoint the simplest case to consider is the circular barrier. Here a separation of variables technique can be used to find a series solution to the first-passage problem.

Firstly it is noted that the FPK equation for $q(a|a_0;t)$ (see equation (43)) may be written in the standard form

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial a}(-m_1 q) + \frac{1}{2}\frac{\partial^2}{\partial a}(m_2 q) \qquad (56)$$

where m_1, the drift coefficient, is given by

$$m_1 = -\frac{-\epsilon^2 F(a)}{\omega_0} - \frac{\pi S_f(\omega_0)}{2a\omega_0^2} \tag{57}$$

and m_2 the diffusion coefficient, is given by

$$m_2 = \frac{\pi S_f(\omega_0)}{\omega_0^2} \tag{58}$$

The solution to equation (56) may be expressed as the eigenfunction expansion (see also equation (1.70))

$$q(a|a_0;t) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \frac{U_i(a)U_i(a_0)}{c_i} w(a) \tag{59}$$

Here $w(a)$ is the stationary solution of the FPK equation ($t \to \infty$), i.e. (see also equation (1.72))

$$w(a) = \frac{k}{m_2} \exp\left[\frac{2}{m_2}\int_0^a m_1(\xi)d\xi\right] \tag{60}$$

and k is a normalisation constant. The eigenfunctions, $U_i(a)$, and the corresponding eigenvalues, λ_i, satisfy the ordinary differential equation

$$\frac{d}{da}\left[\rho(a)\frac{dU_i}{da}\right] + \lambda_i w(a)U_i(a) = 0 \tag{61}$$

where

$$\rho(a) = \frac{1}{2} m_2 w(a) \tag{62}$$

and the boundary condition

$$w(b)U_i(b) = 0 \tag{63}$$

is satisfied. The normalisation constants, c_i, in equation (59) are given by

$$\int_0^b w(a)V_i(a)V_j(a)da = c_i\delta_{ij} \tag{64}$$

In the present problem both $\rho(a)$ and $w(a)$ are positive functions and equation (61) is of the Sturm-Liouville type. It follows that the eigenvalues are real and non-negative, and that $\lambda_1 < \lambda_2 < \ldots$ form a discrete set of values.

If the mean time to failure is long then the first term in equation (59) becomes dominant and $\phi(t|a_0)$ approaches the asymptotic form of equation (4), where here $\alpha = \lambda_1$.

In general an exact analytical solution to equation (63) is very difficult to obtain. However, in the case of a linear oscillator a complete analytical solution can be found in terms of hypergeometric functions. This leads to the following expression for $Q(\tau|A_0)$:

$$Q(\tau|A_0) = \sum_{i=1}^{\infty} D_i \exp(-2\zeta\gamma_i\tau) \tag{65}$$

where

$$D_i = \Phi_i\left[\frac{A_0^2}{2}\right] \frac{\int_0^{\eta^2/2} \Phi_i(x)e^{-x}dx}{\int_0^{\eta^2/2} \Phi_i^2(x)e^{-x}dx} \tag{66}$$

$$\Phi_i(x) = M(-\gamma_i;1;x) \tag{67}$$

$$\eta = b/\sigma \tag{68}$$

and γ_i are the roots, or eigenvalues, of the equation

$$\Phi(\eta^2/2) = 0 \tag{69}$$

Here M( ) is the confluent hypergeometric function. This solution was obtained originally by Helstrom (1959), Rosenblueth and Bustamente (1962) and Gray (1966), using different approaches.

Lennox and Fraser (1974) found that the existing tables were inadequate to evaluate the eigenvalues, γ_i , of equation (69), and numerically solved this equation to obtain the first nine eigenvalues, for $\eta = 1,2$ and 3 . Subsequently Spanos (1980,1982) has discussed efficient algorithms for computing both γ_i and the constants D_i in equation (66), and has presented tabulated and graphical results, for a range of η values.

If the mean time to first passage is large then, as in the general case, equation (65) reduces to the limiting form of equation (4), where here

$$\alpha = 2\zeta\omega_0\gamma_1 \tag{70}$$

The moments, M_n , of the first passage time T , can be obtained directly from the general solution given by equation (65). The result is

$$M_n(A_0) = \sum_{i=1}^{\infty} \frac{D_i n!}{(2\zeta\gamma_i)^n} \tag{71}$$

where time is measured in units of τ . This result can be obtained rather more easily by solving the appropriate GPV equations.

Of most practical interest is the first moment, $M_1(A_0)$ - the mean time to failure - and the variance of the time to failure

$$\mathrm{var}\{T|A_0\} = M_2(A_0) - M_1^2(A_0) \tag{72}$$

$M_1(A_0)$ can be expressed as

$$M_1(A_0) = \frac{1}{2\zeta}\left[\bar{E}_i\left[\frac{\eta^2}{2}\right] - \bar{E}_i\left[\frac{A_0^2}{2}\right] - \ln(\eta^2/A_0^2)\right] \qquad (73)$$

where $\bar{E}_i(\ )$ is the exponential integral function. The variance is given by

$$\text{var}\{T|A_0\} = \frac{1}{\zeta^2}\sum_{i=1}^{\infty}\int_0^{A_0/\sqrt{2}}\frac{(4n-1)\exp(x^2)}{n(2n)!}\,dx \qquad (74)$$

Various alternative ways of expressing these results have been discussed (see Gray (1966)).

In the special case of zero damping, the solution to the PV equation for M_1 may be written as

$$M_1(a_0) = \frac{\omega_0^2}{2\pi S(\omega_0)}\,[b^2-a_0^2] \qquad (75)$$

If the excitation is a white noise, $I = 2\pi S(\omega_0)$ and equation (75) agrees with the exact results given earlier (see equation (38)), since here $E = \lambda_0^2 a^2/2$.

In the more general case of a non-linear oscillator with a circular barrier and an equation of motion given by equation (9), exact solutions for $Q(t|A_0)$ are not available. However, if M_1 is large, useful asymptotic results for this statistic can be deduced from the appropriate PV equation, as Seshadri et al (1980) have shown (see also Lindenberg and Seshadri (1979)). Since $\alpha \to 1/M_1$ as $M \to \infty$, the asymptotic behaviour of M_1 also gives the asymptotic behaviour of $\phi(t|a_0)$.

If the PV equation for M_1 is written in self-adjoint form it can be integrated directly to yield the expression (see Seshadri et al (1980))

$$M_1(a_0) = 2\int_{a_0}^{b} \frac{W(a)}{m_2 w(a)}\, da \tag{76}$$

where

$$W(a) = \int_0^a w(\xi)d\xi \tag{77}$$

For large b the integrals in these equations can be evaluated asymptotically, in specific cases, to yield analytical expressions for M_1 .

For problems involving single or double-sided barriers it is necessary to solve equation (54). If a separable solution

$$q(a|a_0;t) = \Phi(a)\psi(t) \tag{78}$$

is assumed then one finds that (Roberts (1976))

$$\Phi(a) = \lambda\int\Phi(a')p(a|a';\Delta t)da' \tag{79}$$

where

$$\lambda = \psi(t_n)/\psi(t_{n+1}) \tag{80}$$

Thus the general solution to equation (54) may be written as

$$q(a|a_0;t_n) = \sum_{i=1}^{\infty} \Phi_i(a)e^{-\gamma_i t_n} \tag{81}$$

where

$$\gamma_i = \frac{1}{\Delta t} \ln \lambda_i \tag{82}$$

λ_i is the i^{th} eigenvalue of equation (79) and $\Phi_i(a)$ is the corresponding eigenfunction.

In general, analytical solutions of the eigenvalue problem, represented by equation (79), are not available for linear or non-linear oscillators. However, in the limiting case where M_1 becomes very large, the first term of the summation of equation (81) becomes dominant and hence $\phi(t|a_0)$ approaches the asymptotic form of equation (4). Moreover, it can be shown that $\alpha \to \nu$ as $n \to \infty$ (Roberts (1978)). As mentioned earlier, this is probably the correct, exact asymptotic limit for the limiting decay rate.

5.3 Semi-analytic solutions and approximations

One approach to finding solutions in the non-linear case, for the circular barrier, is to seek approximate solutions of the form

$$Q(\tau|A_0) = \sum_{i=1}^{m} c_i(\tau)\Phi_i\left[\frac{A_0^2}{2}\right] \qquad (83)$$

where $\Phi_i(\ )$ are the set of eigenfunctions relating to the linear solution (see equation (67)), $c_i(\tau)$ are functions of time and m is an integer. The expansion of equation (83) is particularly convenient since the eigenfunctions are orthogonal with respect to e^{-x}, and the eigenvalues are known (Spanos (1982)).

By using this Galerkin technique a linear set of differential equations for $c_i(\tau)$ can be determined and solved using standard methods. The resulting estimate of $Q(\tau|A_0)$ will progressively improve in accuracy as the number of terms, m, in the summation is increased. Comparisons with corresponding digital simnulation estimates, for the case of a Van der Pol oscillator, indicate that 5-10 terms suffices for oscillators with a "moderate" degree of non-linearity (Spanos (1982)).

For single or double-sided barriers, an approximate analytical solution to the eigenvalue problem of equation (79) has been given by Mark (1966), for the case of the linear oscillator. For the double-sided barrier is

$$\frac{\alpha}{2} = \text{erf}\left[\left\{ \frac{\eta^2}{2} \tanh(\mu\pi\zeta) \right\}^{\frac{1}{2}} \right] \qquad (84)$$

where erf is the error function. This expression gives the correct asymptotic solution ($\alpha/\nu \to 1$ as $b \to \infty$) .

5.4 Numerical solutions

For a non-linear oscillator, with a circular barrier, the moments $M_1(a_0)$ of the time to first-passage failure can be obtained fairly easily, by numerically solving the sequence of ordinary GPV equations. Results for the first moment have been obtained by this means (Roberts (1976)), for the case of an oscillator with combined linear and quadratic damping. Strictly numerical solutions of the differential equations for the reliability function, $Q(t|a_0)$ do not appear to have been attempted.

For single and double-sided barriers, equation (54) is the governing equation. As mentioned earlier, this can be solved numerically by marching in time, with steps Δt , provided that $p(a|a_0;\Delta t)$ is known analytically. An eigenfunction expansion for this transition density function can be written, which is of the same form as equation (59) (here the unconditional transition density function is required, so $b \to \infty$) . However, for the class of oscillators under discussion an analytical solution for the eigenfunction problem is known only for the case of linear damping. This solution leads to the expression (see also Chapter 3)

$$p(a|a_0;\Delta t) = \frac{a}{\beta}\exp\left\{-\frac{(a^2+a_0^2 e^{-2\zeta\omega_0\Delta t})}{2\beta}\right\} \times I_0\left[\frac{aa_0 e^{-2\zeta\omega_0\Delta t}}{\beta}\right] \tag{85}$$

where

$$\beta = \sigma^2[1 - \exp(-2\zeta\omega_0 t)] \tag{86}$$

Equation (54) has been solved numerically, for the case of a linear oscillator, by performing numerical integrations at each time step, and using equations (81) and (82) (Roberts (1976)). It was found, as the general theory predicts (Miklin (1957)) that $q(a|a_0;t_n)$ becomes proportional to the first eigenfunction, $\phi_1(a)$, as n becomes large, and that

$$R(n) = \frac{Q(n)}{Q(n+1)} \rightarrow \lambda_1 \tag{87}$$

as $n \rightarrow \infty$, where

$$Q(n) = \int_0^b q(a|a_0;t_n)da \tag{88}$$

Thus the first eigenvalue (and hence the limiting decay rate) can be estimated by marching in time, until the ratio $R(n)$ reaches its limiting value. For $\eta < 3$ this technique is satisfactory but for higher barrier levels λ_1 is so close to unity that the degree of computational effort required to produce accurate estimates of α is prohibitive. This difficulty can be overcome by expanding $q(a|a_0;t)$ in terms of the eigenfunctions of $p(a|a_0;t)$, which are in the form of Laguerre polynomials (see Chapter 3). This technique leads to a recurrence relationship for the unknown coefficients in the expansion, which are easily solved (Roberts (1976)).

Results have been presented for $P(t|a_0)$ and α, for a range of barrier levels, η and damping factors ζ, and compared with corresponding results obtained by using the circular barrier. Similar results for α were reported earlier by Mark (1966).

Lutes et al (1980) have fitted empirical expressions to the results for α given by Roberts (1976) to allow easy extension to other values of η and ζ. The best simple expression they found, for the double-sided barrier, was

$$\frac{\alpha}{\nu} = 1 - 1.075\left[\eta \exp\left[-\frac{\nu}{2}\right]\right]^{w} \tag{89}$$

where

$$w = 0.2364 + 28.14q^2 \tag{90}$$

and q is a measure of the bandwidth of the response process, given by

$$q^2 = 1 - (1-\zeta^2)^{-1}\left[1 - \frac{1}{\pi}\tan^{-1}\left\{\frac{2\zeta(1-\zeta^2)^{\frac{1}{2}}}{1-2\zeta^2}\right\}\right]^2 \tag{91}$$

For small ζ this can be approximated by

$$q^2 \sim \frac{4}{\pi}\zeta(1-1{\cdot}1\zeta) \tag{92}$$

For oscillators with non-linear damping a discrete random walk analogue, $R(t_j)$, for the process $a(t)$ can be constructed, as shown in Chapter 3. For $R(t_j)$ the appropriate discrete equivalent to equation (54) is

$$Q(a_k,t_{n+1}) = Q(a_{k-1},t_n)r_{k-1} + Q(a_{k+1},t_n)q_{k+1} \tag{93}$$

where $Q(a_k,t_n)$ is the probability of being in state a_k,t_n, *without* exceeding the barrier. This results enables $Q(a_k,t_n)$ to

be found by marching in time steps, δt . When $t_n = i\Delta t$, where Δt is the interval between the vertical lines of the approximated single or double-sided barrier (see Fig. 4.2(c)) then the probability mass is absorbed, for $a_k > b$, and does not appear in the subsequent diffusion process.

The random walk analoque has been used to estimate $P(t|a_0)$, and the limiting decay rate, α , for oscillators with combined linear and cubic damping (Roberts (1978)). The stability and robustness of the numerical scheme was found to be remarkably good, with no difficulty in obtaining results for barrier levels as high as $\eta = 6$, in the linear case.

5.5 Use of the energy envelope

It was shown earlier, in Chapter 3, that for oscillators with the equation of motion

$$\ddot{x} + \epsilon^2 h(x,y) + g(x) = \epsilon z(t) = f(t) \qquad (94)$$

the energy envelope $E(t)$ could be approximated accurately as a one-dimensional Markov process, as $\epsilon \to 0$. The conditional transition density function $q(E|E_0;t)$ for the Markov model of $E(t)$ is governed by the same FPK equation as the unconditional transition density function $p(E|E_0;t)$. Thus (see equations 3.120)

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial E}\left[\left\{ \epsilon^2 B(E) - \frac{I}{2} \right\} q \right] + \frac{I}{2} \frac{\partial^2}{\partial E^2}[C(E)q] \qquad (95)$$

where $B(E)$ and $C(E)$ are the functions defined previously, and $I = 2\pi S_f(0)$.

The backward operator, $\mathscr{L}$, for $E(t)$ is

$$\mathscr{L} = -\left[\epsilon^2 B(E_0) - \frac{I}{2}\right]\frac{\partial}{\partial E_0} + \frac{I}{2}C(E_0)\frac{\partial^2}{\partial E_0^2} \qquad (96)$$

The easiest first-passage problem to solve, when dealing with $E(t)$, is to find the probability, $Q(t|E_0)$ that $E(t)$ stays below a critical level, E, in the interval 0-t, given that $E(0) = E_0$, a known initial value. This is a natural generalisation of the circular barrier problem considered earlier, for the special case where $g(x)$ is linear. If the damping is sufficiently light then $Q(t|E_0)$ will be a good approximation to the probability that $x(t)$ stays below the level b, in the interval 0-t, where $E = V(b)$.

A complete analytical solution for $Q(t|E_0)$ can, in principle, be obtained by using an eigenfunction expansion, similar to that given previously for $a(t)$ (see equation (59)). However, the appropriate eigenfunction problem has been solved, so far, only for case of linear damping and a power-law spring - i.e.

$$g(x) = k|x|^{\nu}\mathrm{sgn}(x) \qquad (97)$$

and k and ν are constants. In this case the functions $B(E)$ and $C(E)$ are both proportional to E (see equations (3.128) and (3.129)), so the eigenfunction problem is not dissimilar from that in the linear case. This solution for $Q(t|E_0)$ (Roberts (1976)) generalises the result given earlier for the linear case where $\nu = 1$ (see equations (65) to (69)).

The moment, $M_n(E_0)$, of the time it takes for $E(t)$ to first reach E, starting from E_0, can be determined fairly easily by solving the GPV equations recursively, for $n = 1,2,...$ etc. (Roberts (1976)). In the case of a linear damper and a power-law spring, the moment may be found analytically. In other cases the GPV equations may be solved numerically, without difficulty, since they are ordinary differential equations. Specific results have been obtained for the Duffing oscillator, with linear damping (Roberts (1976)).

In the special case of zero damping ($\epsilon^2 = 0$) the PV equation for $M_1(E_0)$ becomes

$$\frac{I}{2}\frac{dM_1}{dE_0} + \frac{I}{2}C(E_0)\frac{d^2M_1}{dE_0^2} = -1 \qquad (98)$$

By inspection the solution (subject to the appropriate boundary conditions) is in complete agreement with the exact solution given by equation (38).

The diffusion equations for $E(t)$ may be used to find approximate solutions to the single and double-sided barrier problems for the displacement process $x(t)$. This involves a generalisation of the arguments given earlier, for $a(t)$, and leads to the following discrete-time diffusion equation:

$$q(E|E_0;t_{n+1}) = \int_0^h p(E|E';\Delta t)q(E'|E_0;t_n)dE' \qquad (99)$$

where $h = V(b)$. This is clearly a generalisation of equation (54) for $a(t)$. $q(E|E_0;t_n)$ is the probability that $E(t)$ reaches the differential element centred at E, at time t_n, *without* intersecting the vertical barrier lines, spaced Δt apart.

In the case of a non-linear spring the choice of an appropriate value for Δt in equation (99) is not obvious. It has been shown (Roberts (1978)) that it is best to choose as follows:

$$\Delta t = \mu T(h) \qquad (100)$$

where, as before, $h = V(b)$ and $\mu = 1$ for single-sided barriers, 1/2 for double-sided barriers. $T(E)$ is the undamped natural period of the oscillator (see equation (1.40)). With this choice of Δt it can be shown that $\alpha \to \nu$ as $b \to \infty$.

Equation (99) can be marched in time, with step length Δt, provided that the unconditional transition density function is known. Available analytical solutions for this function are limited to the class of oscillators with linear damping and a non-linear spring of power-law form (see Chapter 3). Estimates of the limiting decay rate, for some oscillators in this class, are given by Roberts (1976). These were obtained by numerically evaluating the evolution of $q(E|E_0;t)$, according to equation (99). For high barrier levels an expansion technique for $q(E|E_0;t)$ is required, to ensure numerical stability (see Roberts (1976)).

In cases where $p(E|E_0;t)$ is not known analytically a random walk analogue of the diffusion equation can be used, which is a generalisation of the analogue discussed earlier, for the case where $g(x)$ is linear. This method is discussed in detail by Roberts (1978), where results are presented for the case of a Duffing oscillator with linear damping. Alternatively, an implicit finite difference approximation can be used to solve the FPK equation governing $q(E|E_0;t)$ (Roberts (1986)). As with the random walk analogue, this enables one to march the diffusion process forward in time, with steps, δt. However, the latter approach has the advantage that δt is no longer necessarily proportional to the square of the amplitude increment. Thus larger steps in time can be used, without sacrifice in accuracy.

5.6 Oscillators with parametric excitation

The method of stochastic averaging can also be used to study the effect of parametric excitation on first-passage times (Dimentberg and Sidorenko (1978)), Ariaratnam and Tam (1979)).

Consider, for an example, an oscillator with combined external and parametric excitation, with an equation of motion of the form

$$\ddot{x} + \omega_0[2\zeta+g(t)]\dot{x} + \omega_0^2 x[1+h(t)] = f(t) \qquad (101)$$

Here $g(t)$, $h(t)$ and $f(t)$ are stationary stochastic processes with zero means, and $g(t)$ and $h(t)$ are parametric excitations. Here the parametric excitation is more general than in the oscillator considered earlier, in Chapter Two; it reduces to the earlier type if $g(t) = 0$.

On applying the method of stochastic averaging, on the assumption that the damping is light, the following appropriate stochastic differential equation for the amplitude process, $a(t)$, may be derived (Ariaratnam and Tam (1979), Roberts (1982)):

$$\dot{a} = (-\alpha a + \frac{\beta}{2a}) + (\gamma a^2 + \beta)^{\frac{1}{2}} \xi(t) \tag{102}$$

where

$$\alpha = \zeta\omega_0 - \frac{\pi\omega_0^2}{8} [2S_g(0) + 2S_g(2\omega_0) + 3S_h(2\omega_0) + 6\Psi_{hg}(2\omega_0)] \tag{103}$$

$$\beta = \frac{\pi S_f(\omega_0)}{\omega_0^2} \tag{104}$$

and

$$\gamma = \frac{\pi\omega_0^2}{4} [2S_g(0) + S_g(2\omega_0) + S_h(2\omega_0) + 2\Phi_{hg}(2\omega_0)] \tag{105}$$

Here $S_g(\omega), S_h(\omega)$ and $S_f(\omega)$ denote, respectively, the power spectra of $g(t), h(t)$ and $f(t)$, whereas $\Phi_{hg}(\omega)$ and Ψ_{hg} are the real and imaginary components of the cross-spectrum of $h(t)$ and $g(t)$ - i.e.

$$\Phi_{hg}(\omega) + i\Psi_{hg}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E\{h(t)g(t+\tau)\}e^{-i\omega\tau} d\tau \tag{106}$$

The FPK and backward Kolmogorov equations follow immediately from equation (102). Thus the backward operator is

$$\mathscr{L} = (-\alpha a_0 + \frac{\beta}{2a_0})\frac{\partial}{\partial a_0} + \frac{1}{2}(\gamma a_0^2+\beta)\frac{\partial^2}{\partial a_0^2} \qquad (107)$$

By solving the differential equation for $\phi(t|a_0)$, with appropriate boundary conditions, Ariaratnam and Tam (1979) have obtained an analytical solution for the probability that $a(t)$ stays below a specified level for a given time. This result, in terms of the hypergeometric function F_1 , is a generalisation of the result given earlier in this Chapter, for a linear oscillator with purely external excitation.

If only the moments of the first passage time, T , are required then they may be obtained directly from the GPV equations, as previously noted. Results of this kind have been given by Dimentberg and Sidorenko (1978) and by Ariaratnam and Tam (1979)).

5.7 Non-stationary problems

The case of purely external excitation will now be returned to. If this excitation is considered to be a modulated white noise

$$f(t) = \alpha(t)\xi(t) \qquad (108)$$

where $\alpha(t)$ is a deterministic modulating function and $\xi(t)$ is stationary white noise, of unit strength, then the "zero-start" solutions discussed earlier in this paper relate to the case where $\alpha(t)$ is a step function.

If the approximate energy envelope is adopted then, if $\alpha(t)$ is some arbitrary function, I in the diffusion equation for $E(t)$ can be replaced by $\alpha^2(t)$. Numerical solutions for the

diffusion equations can easily be generalised to allow for time-dependency of $\alpha^2(t)$. For example, Roberts (1976) used the discrete-time envelope method to obtain estimates of the probability of first passage failure of a linear oscillator, where

$$\alpha(t) = A[\exp(-c_1 t) - \exp(-c_2 t)] \quad ; \quad t > 0 \qquad (109)$$

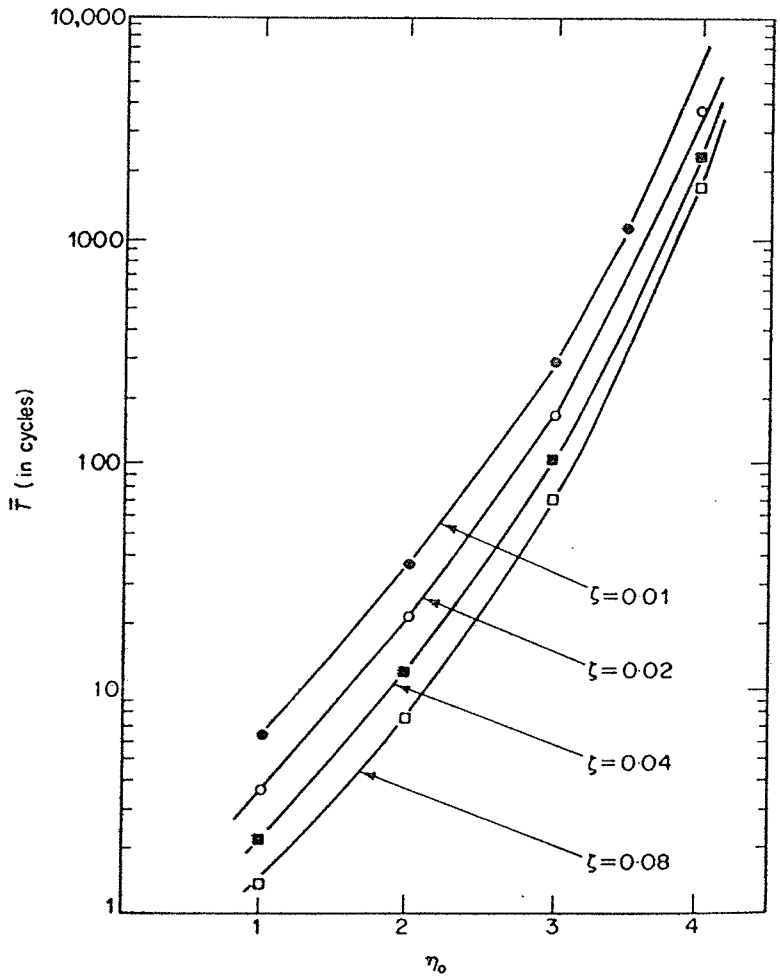
and A, c_1 and c_2 are constants.

In cases where the stochastic averaging method is applicable (i.e. g(x) linear, or nearly so) a rather more general model of the non-stationary excitation, f(t) , can be adopted, using the concept of the evolutionary power spectrum (see Chapter Three). Using this spectral representation one simply replaces $S(\omega_0)$ by $S(\omega_0,t)$, in the differential generator $\mathscr{L}$, for example (see equation (44)).

Whilst numerical solutions are easily modified to deal with modulated excitation, analytical solutions are much more difficult. One semi-analytical approach is to use step-function modulated excitation as the basis for a Galerkin method of solution. Results for a linear oscillator with modulated excitation, and a circular barrier, have been obtained recently by Spanos and Solomos (1983).

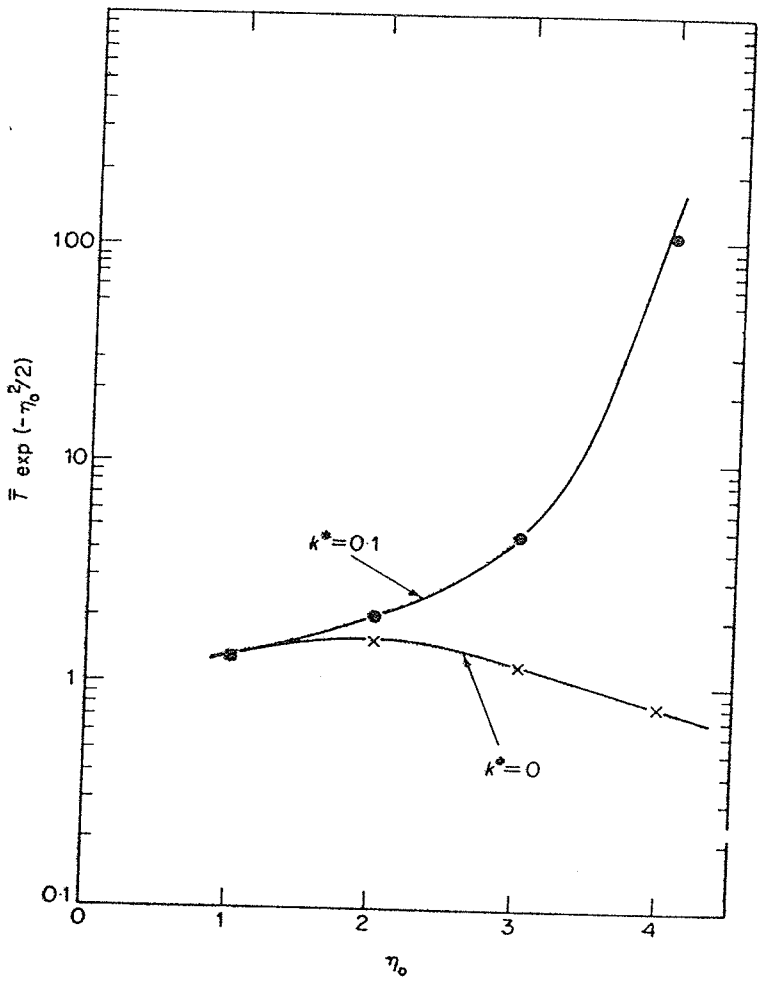
6. Conclusions

For oscillators with light damping methods based on the use of stochastic averaging offer a relatively simple approach. The reduction in dimensionality of the relevant Markov process, from two to one, results in very considerable simplifications, with a negligible loss in accuracy. Comparisons between the result obtained by numerically solving the "exact" two-dimensional diffusion equations, and the corresponding results obtained from the approximate one-dimensional diffusion equations (see Figs. 4.3 to 4.6, where the mean time to failure , denoted $\bar{T}$, is plotted against barrier height for various types of oscillator



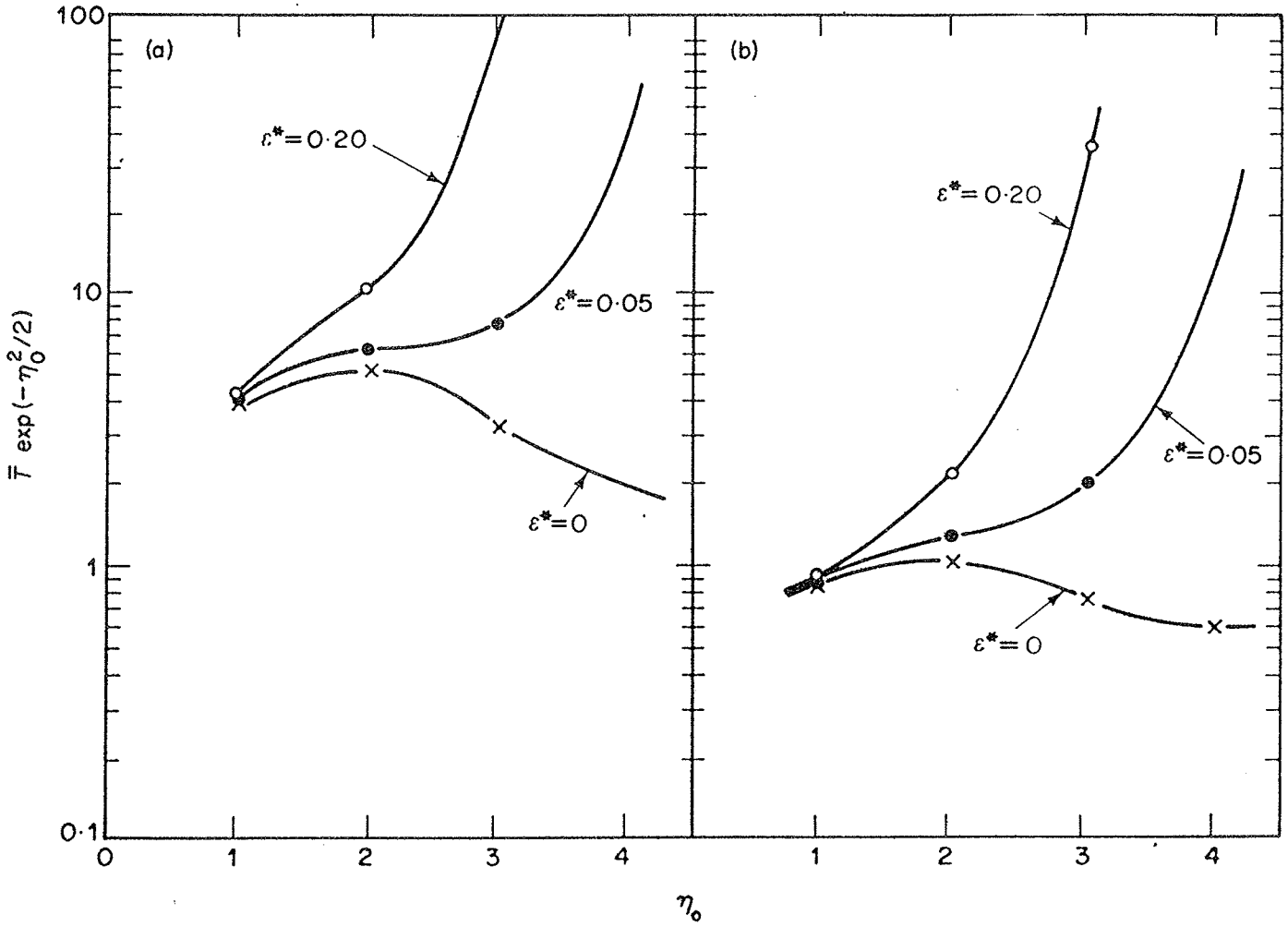
The linear oscillator. Variation of $\bar{T}$ (in cycles) with barrier height, η_0, for various values of ζ ——, Present theory; finite element solution: ●, $\zeta = 0{\cdot}01$; ○, $\zeta = 0{\cdot}02$; ■, $\zeta = 0{\cdot}04$; □, $\zeta = 0{\cdot}08$.

Fig. 4.3.



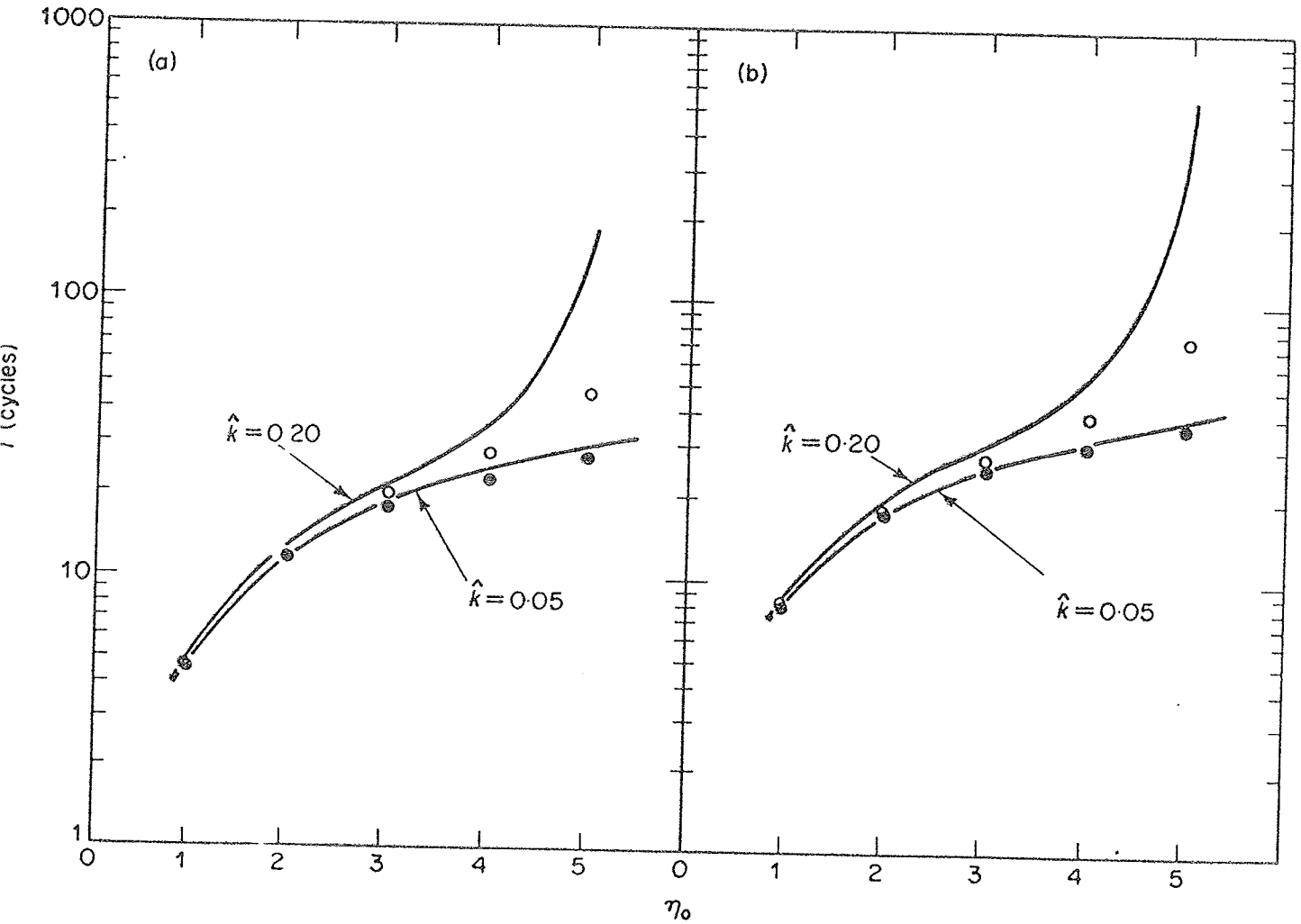
The oscillator with non-linear damping and linear stiffness. Variation of $\bar{T}\exp(-\eta_0^2/2)$ with η r $k^*=0$ and 0·1. ——, Present theory; finite element solution: ×, $k^*=0$; ●, $k^*=0·1$.

Fig. 4.4.



The Duffing oscillator with linear damping. Variation of $\bar{T}\exp(-\eta_0^2/2)$ with η_0. $\varepsilon^* = 0, 0{\cdot}05$ $0{\cdot}20$. (a) $\zeta = 0{\cdot}01$; (b) $\zeta = 0{\cdot}08$. ——, Present theory; finite element solution: ×, $\varepsilon^* = 0$; $\varepsilon^* = 0{\cdot}05$; ○, $\varepsilon^* = ($

Fig. 4.5.



The van der Pol oscillator with linear stiffness. Variation of $\bar{T}$ with η_0. $\hat{k}=0{\cdot}05$ and $0{\cdot}20$. (a) $=0{\cdot}01$; (b) $\zeta=0{\cdot}08$. ——, Present theory; finite element solution: ●, $\hat{k}=0{\cdot}05$; ○, $k=0{\cdot}20$.

Fig. 4.6.

(see Roberts (1986(b)) for details) indicate that the latter are certainly sufficiently accurate for engineering purposes. This conclusion is reinforced by comparisons with simulated data and the fact that exact solutions are obtained in the case of no damping.

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AVERAGING METHODS IN RANDOM VIBRATION

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CHAPTER FIVE

Application of the Stochastic Averaging Method to Specific Engineering Problems

1. Introduction

In this concluding Chapter the application of the stochastic averaging method to the following two specific engineering problems will be discussed.

(1) The rolling motion of a ship in irregular beam seas.
(2) The response of a hysteretic oscillator to wide-band random excitation.

In both cases the energy envelope method, as developed in Chapter Three, will be employed to yield information on response statistics such as the standard deviation, and the distribution of amplitude.

2. Roll motion of a ship in random waves

In assessing the possibility of a ship capsizing, when operating in a particular sea state, attention must be paid to the probability of the rolling motion reaching large, and potentially dangerous amplitudes. Rolling motion can become excessive when the natural roll frequency lies in the frequency range for which wave energy is dominant. In these circumstances the roll response can be regarded as a resonance phenomenon, in which the

amplitude of the roll motion is critically dependent on the magnitude of the hydrodynamic damping which is present. Unfortunately the damping in the roll mode is often very light with the result that large roll amplitudes can readily occur under resonant conditions.

Here it is shown that the stochastic averaging method, based on the energy envelope, can be brought to bear on this problem, provided that attention is restricted to the case of a ship rolling in unidirectional beam waves. In these circumstances it is possible to justify the adoption of a single degree of freedom roll model, in which non-linearities are present in both the damping and restoring moment terms.

The theoretical results obtained are validated through a comparison with experimental results obtained from a model ship in a wave tank.

2.1 Equation for roll motion

The theory which will be studied here is based on the assumption that the roll motion of a ship in beam waves can be modelled by a non-linear, second order differential equation of the form,

$$I\ddot{\phi} + \epsilon^2 C(\dot{\phi}) + K(\phi) = M(t) \qquad (1)$$

where

I = roll inertia

ϕ = angle of roll

$C(\dot{\phi})$ = non-linear damping moment

$K(\phi)$ = non-linear restoring moment

$M(t)$ = roll excitation moment

ϵ^2 = a scaling parameter (ϵ^2 small)

The adoption of this model implies that the roll motion can be uncoupled from the other motions, such as sway, by a suitable choice of coordinates. Some justification for this uncoupled equation of roll, will be given shortly. It is observed that equation (1) can be simplified somewhat by dividing throughout by I . Thus

$$\ddot{\phi} + \epsilon^2 F(\dot{\phi}) + G(\phi) = X(t) \tag{2}$$

where

$$F = C/I, \quad G = K/I, \quad X = M/I \tag{3}$$

2.2 Application of the energy envelope method

As before, if the energy envelope, $E(t)$ is defined as

$$E(t) = \frac{\dot{\phi}^2}{2} + V(\phi) \tag{4}$$

where

$$V(\phi) = \int_0^{\phi} G(\xi)\,d\xi \tag{5}$$

then, if ϵ^2 is sufficiently small, $E(t)$ can be modelled as a one-dimensional Markov process. The transition density function, $p(E|E_0;t)$ for $E(t)$ is then given by (see Chapter 3)

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial E}[m(E)p] + \frac{1}{2}\frac{\partial^2}{\partial E^2}[D(E)p]$$

where expressions for $m(E)$ and $D(E)$ are given in Chapter Three by equations (3.144) and (3.145).

If the excitation is not too severe, then it is possible to consider, in an appropriate sense, that the response $E(t)$ will

reach stationarity. (However, it is noted that, strictly speaking, it is improper to discuss the stationary response distribution in the roll mode, since stationarity will never actually be reached). Setting $\partial p/\partial t = 0$ one obtains, as before, the stationary density function

$$w(E) = \frac{C}{D(E)} \exp\left\{2 \int_0^E \frac{m(\xi)}{D(\xi)} d\xi\right\} \tag{6}$$

where C is a normalisation constant.

From equation (6) one can deduce various response statistics. For example, the amplitude process, $a(t)$, where

$$E = V(a) \tag{7}$$

has the stationary density function

$$w(a) = w(E) \frac{dE}{da} = G(a)w(E) \tag{8}$$

Moreover, as noted earlier, the joint stationary density function of ϕ and $\dot{\phi}$ is given by

$$w(\phi,\dot{\phi}) = \frac{w(E)}{T(E)} \tag{9}$$

where $T(E)$ is the period of free oscillations, without damping. From this one can obtain expressions for many statistics associated with ϕ and $\dot{\phi}$. In particular, one finds, by integration, the standard deviation, σ , of the roll response, $\phi(t)$. Thus

$$\sigma^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^2 w(\phi,\dot{\phi}) d\phi \tag{10}$$

and, from equations (9) and (10)

$$\sigma^2 = \int_0^\infty w(E) D_2(E) dE \tag{11}$$

where

$$D_2(E) = \frac{2\sqrt{2}}{T(E)} \int_{-\infty}^{\infty} \frac{\phi^2 d\phi}{[E-V(\phi)]^{\frac{1}{2}}} \tag{12}$$

For most ships, undergoing moderate roll motion, (i.e., not close to capsize) the effect of non-linearity in the restoring moment is relatively weak. In this case, as pointed out in Chapter 3, the energy envelope theory indicates that the shape of the input spectrum does not play a role in the determination of $m(E)$ and $D(E)$. Thus, for linear, or near-linear restoring moments, the basic assumptions of the proposed theory are effectively equivalent to making a white noise approximation for $x(t)$, with constant spectral level, $S_x(\omega_0)$.

Since in the present application the spectrum of $x(t)$ is decidedly non-white (as will be shown later) it is appropriate to apply the modified theory, outlined in Chapter 3. This involves scaling the spectrum $S_x(\omega)$, in the expressions for $m(E)$ and $D(E)$ with a factor $r(\zeta)$, derived from linear theory (see Chapter 3 section 4.7).

2.3 Theoretical modelling

In the experimental tests, to be discussed later, a scale model of the Fishing Protection Vessel the "Sulisker" was used. We now describe how the general theory, outlined earlier, can be applied to this particular ship.

2.3.1 Coupling with other motions

As a first step it is necessary to validate the uncoupled roll equation of motion given by equation (1). A complete, non-linear analysis, with all six degrees of freedom represented, is not feasible analytically; in any event the parameters, and even the form, of the general equations of motion are uncertain. However, a guide to the validity of equation (1) can be obtained by considering small displacements and using a linearised theory.

A comprehensive linear analysis was carried out using a computer programme known as NMIWAVE (Standing (1979)). As a first stage in the calculation, the hydrodynamic forces and moments acting on the ship were computed, assuming that the ship was subjected to unit amplitude, harmonic beam waves of various frequencies. The next stage involved computing the elements of the added-mass matrix, at various frequencies, by assuming that the ship oscillated harmonically in calm water. This calculation was carried out for each degree of freedom. Finally, the overall transfer functions between the wave input and the various components of ship displacement were calculated by combining the computed force vector information with the parameters of the general linear equations of motion for six degrees of freedom. For a specified wave elevation spectrum, $S_W(\omega)$, one could then use the overall transfer function to compute the spectra, and cross-spectra, of the six displacement coordinates and hence statistics such as the standard deviations of the displacement components.

The calculations were carried out with the origin of the coordinate system located at the centre of mass of the ship. A study of the computed added-mass matrix revealed that, with this reference point for the motions, there is significant coupling between *sway* and *roll*. The other motions were much less strongly coupled with roll. This point was further demonstrated by running the NMIWAVE programme with the inertias relating to displacement other than roll and sway set artificially at a very high level, so that the number of degrees of freedom was effectively reduced

to two (corresponding to roll and sway). The standard deviations of the roll displacement in this case were within 1% of the results obtained with the fully coupled, six-degree-of-freedom equations.

On the assumption that only sway motion is coupled with roll, and that the influence of the sway damping term on the roll motion is small, it is possible to decouple the roll equation by moving the origin of the coordinate system to a new position, called the "roll centre", a distance z vertically below the centre of mass of the ship, as shown in Fig. 5.1. The appropriate expression for z is

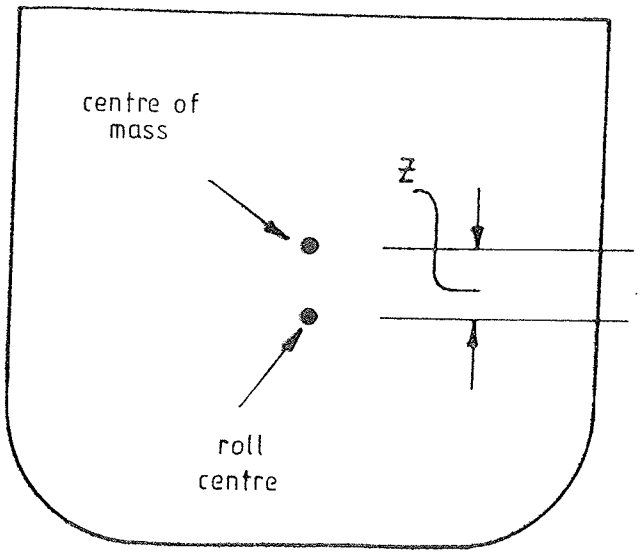


Fig. 5.1.

$$z = \frac{A_{21}}{A_{11}} \tag{13}$$

where A_{21} is the off-diagonal added-mass component and A_{11} is the total mass (including added mass) of the sway motion. The effect of moving the coordinate system is to change the values of the roll inertia (slightly) and the roll moment (significantly). Calculations based on the resulting single-degree-of-freedom equation were found to give standard deviation values for roll

displacement within 10% of the results obtained with the fully coupled six-degrees-of-freedom equation.

It can be concluded that, at least for small, linearized motions, the roll can be modelled as a single-degree-of-freedom equation, provided that the coordinate origin is chosen at the roll centre. In the absence of further information, this roll centre is also the best origin to adopt in the case of larger, non-linear motions.

2.3.2 Restoring moment

The exact hydrostatic restoring moment of the Sulisker, expressed as the variation of the righting lever (see Chapter One) with roll angle, in calm water, is shown in Fig. 5.2. For analysis purposes it is useful to approximate this function by following analytical expression

$$G(\phi) = \omega_0^2 \phi (1 + \lambda \phi^2) \qquad (14)$$

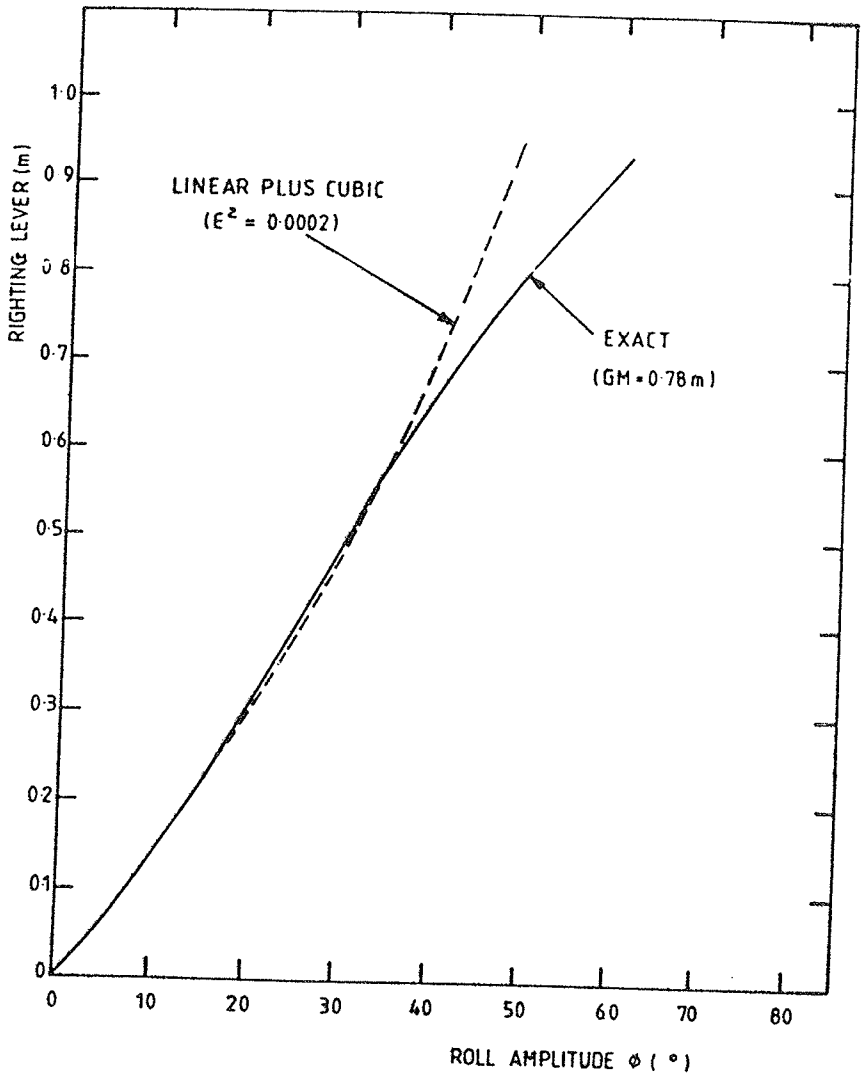
where (with ϕ in degrees) $\lambda = 0.0002$ and ω_0 is the undamped frequency of roll oscillation.

2.3.3 Damping moment

Free decay tests, carried out on a scale model of the Sulisker, were analysed by the method described in Chapter One. The analysis showed that the damping is well represented by a linear-plus-quadratic form - i.e.

$$F(\dot{\phi}) = a\dot{\phi} + b\dot{\phi}|\dot{\phi}| \qquad (15)$$

where a and b are constants.



Restoring moment characteristic

Fig. 5.2.

2.3.4 The complete equation of motion

On combining expressions one has the following equation of motion:

$$\ddot{\phi} + \epsilon^2[a\dot{\phi} + b\dot{\phi}|\dot{\phi}|] + \omega_0^2\phi(1 + \lambda\phi^2) = x(t) \qquad (16)$$

with the introduction of the non-dimensional time

$$\tau = \omega_0 t \tag{17}$$

equation (16) can be written as

$$\ddot{\phi} + 2(a^*\dot{\phi} + b^*\dot{\phi}|\dot{\phi}|) + \phi(1 + \lambda\phi^2) = \frac{x(\tau)}{\omega_0^2} \tag{18}$$

where differentiation is now with respect to τ and a^* and b^* are non-dimensional damping coefficients, defined by

$$a^* = \frac{\epsilon^2 a}{2\omega_0} \quad , \quad b^* = \frac{\epsilon^2 b}{2} \tag{19}$$

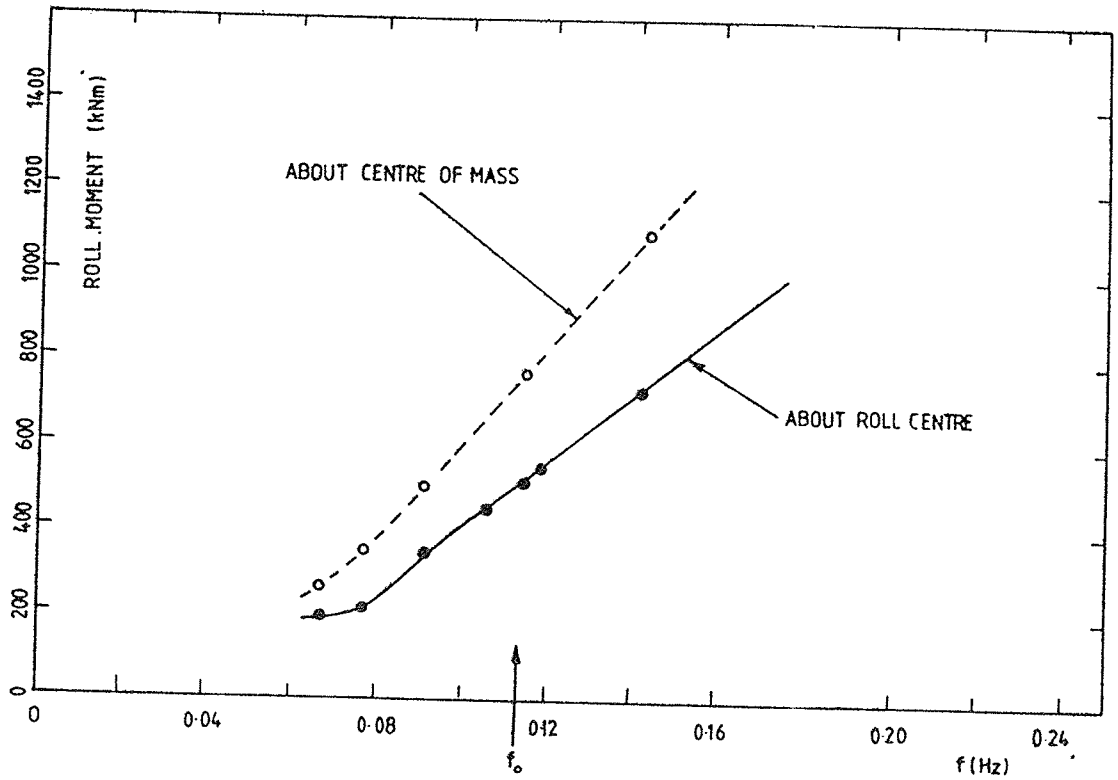
From free-decay data one finds that

$$a^* = 0.0107 \quad , \quad b^* = 0.00474 \tag{20}$$

2.3.5 Wave moment spectrum

The wave elevation spectrum, $S_w(\omega)$ must be converted into a wave moment spectrum, $S_M(\omega)$, and hence to the spectrum of $x(t)$, denoted $S_x(\omega)$.

Here the simplifying approximation is made that the linearised analysis, referred to earlier, can be used to convert $S_w(\omega)$ to $S_M(\omega)$. This is tantamount to assuming that, even for fairly large angles of roll, the wave moment experienced by the ship is not dependent on the actual motion of the ship. Such a decoupling is only strictly correct for small, linearised motions and the consequences of adopting this assumption in the analysis can only be assessed through a comparison between the subsequent theoretical predictions and experimental results.



Variation of roll moment with frequency for unit amplitude harmonic beam waves

Fig. 5.3.

Thus if $M(\omega)$ is the magnitude of the roll moment experienced by the ship in harmonic beam waves, of frequency ω and unit height, as computed from linear theory (NMIWAVE programme) it will be assumed that

$$S_M(\omega) = M^2(\omega) S_W(\omega) \qquad (21)$$

The variation of $M(\omega)$ with frequency will depend on the choice of origin, as explained previously. Here the roll centre will be used, rather than the mass centre of the ship, for the reasons given earlier. Fig. 5.3 shows the variation of $M(\omega)$ with frequency for moments about the roll centre, and about the centre of mass. It is noted that the result of shifting the origin of the coordinate system is to substantially reduce the value of $M(\omega)$.

To convert $S_M(\omega)$ to $S_x(\omega)$, a knowledge of the roll inertia is required, The added-mass component of the roll inertia is frequency-dependent, but varies slowly with frequency. Since the rolling motion takes place predominantly at the natural frequency of oscillation, a good approximation is to assume that the total roll inertia is independent of frequency.

The measured natural roll frequency (at low amplitudes) in free decay tests was about 0.113 Hz (converted to full scale). The added roll inertia at this frequency is 2.60×10^3 tonne(t) m^2 . Adding this to the roll inertia of the (dry) ship gives a total roll inertia of 21.2×10^3 t m^2 . This value of I gives a theoretical natural roll period which is very close to that measured experimentally.

2.4 Computing the roll distribution

To evaluate m(E) and D(E) the approximations given in equations (3.146) and (3.147) are adequate. Thus

$$m(E) = -\epsilon^2 B(E) + \frac{\pi}{2}(s_1^2 + c_1^2)\, S_x[\omega(E)] \tag{22}$$

$$D(E) = 2\pi\, E\, s_1^2\, S_x[\omega(E)] \tag{23}$$

2.4.1 Damping function B(E)

With time measured in units of τ it can be shown that (Roberts (1985))

$$\epsilon^2 B(E) = 2[a^* E\, \alpha(E) + b^* E^{3/2} \beta(E)] \tag{24}$$

where, over the range of values of concern here $(0 < |\phi| < 35^o)$ the following asymptotic approximations are valid

$$\alpha(E) = 1 + 0.375\ m \tag{25}$$

$$\beta(E) = 1.20(1 + 0.450\ m) \tag{26}$$

where

$$m = \frac{\lambda A^2}{2(1 + \lambda A^2)} \tag{27}$$

and A is the amplitude of roll, such that

$$E = V(A) \tag{28}$$

The damping function $Q(E)$ which is needed to use the modified theory is given by

$$Q(E) = a^* \alpha(E) + b^* E^{1/2} \beta(E) \tag{29}$$

2.4.2 Fourier coefficients s_1 and c_1

To evaluate s_1 and c_1 it is necessary to solve the free undamped oscillation equation. One finds to a good approximation (see Roberts (1982)) that

$$s_1 = 1 + \frac{3}{16}\ m \tag{30}$$

with an error of order m^2. Similarly, for small m,

$$c_1 = 1 - \frac{3}{16}\ m \tag{31}$$

On combining, and neglecting terms of order m^2 again, one has

$$s_1^2 + c_1^2 = 2 \tag{32}$$

2.4.3 Evaluation of $\omega(E)$

From the solution for free undamped oscillations, one finds that the period of oscillation, $T(E)$, measured in units of τ is given by

$$T(E) = \frac{4K(m)}{(1 + \lambda A^2)^{\frac{1}{2}}} \tag{33}$$

where $K(m)$ is the complete elliptic integral. Hence

$$\omega(E) = \frac{2\pi}{T(E)} \tag{34}$$

2.4.4 Evaluation of $D_2(E)$

$D_2(E)$ may be evaluated from equation (12). For a linear-plus-cubic restoring moment one finds that

$$D_2(E) = A^2 \frac{M(m')}{L(m')} \tag{35}$$

where

$$M(m') = \int_0^{\pi/2} \frac{\sin^2\theta d\theta}{(1 + m'\sin^2\theta)^{1/2}} \tag{36}$$

$$L(m') = \int_0^{\pi/2} \frac{d\theta}{(1 + m'\sin^2\theta)^{1/2}} \tag{37}$$

and

$$m' = \frac{\lambda A^2}{2+\lambda A^2} \tag{38}$$

$D_2(E)$ has the dimensions of E, and $D_2(E) \to 0$ as $E \to 0$.

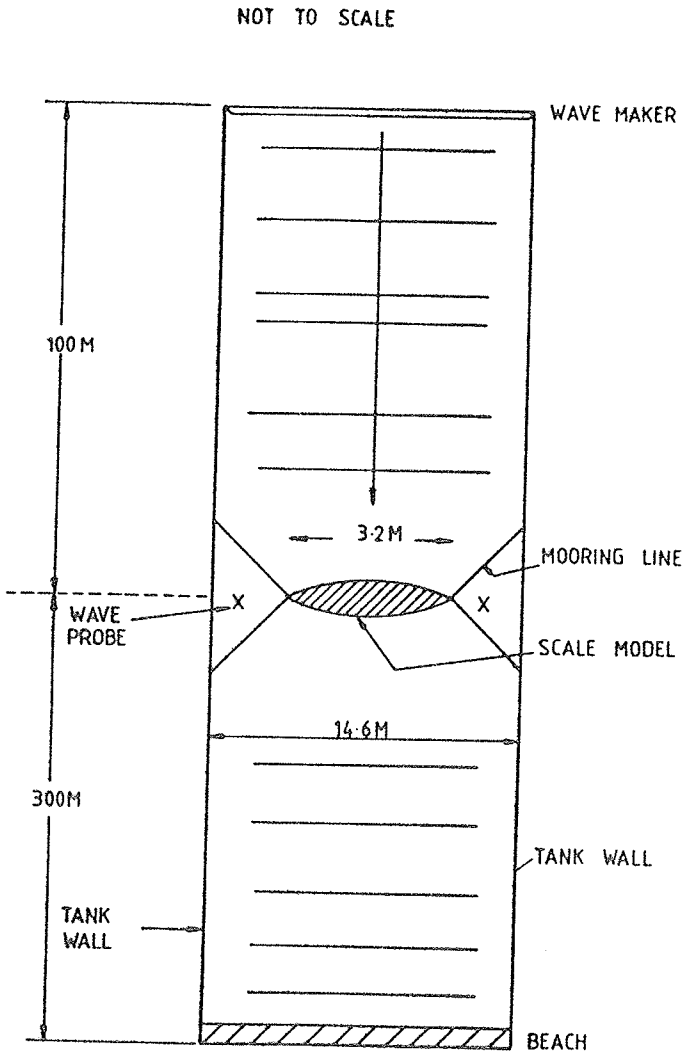


Fig. 5.4.

2.5 Experimental validation

The tests to be discussed here were carried out in a 1:20 scale model of the Sulisker to prevent the model drifting down the tank it was partially restrained by a mooring arrangement, as shown in Fig. 5.4. Irregular unidirectional waves were generated by a wavemaker at one end of the tank and the model was positioned for beam wave encounter.

With the wavemaker set to produce waves with a prescribed "target" spectrum, simultaneous measurements of roll motion and wave elevation, in the vicinity of the model, were recorded. Long, non-repeating runs were necessary to generate sufficient roll data for the statistics of roll response to be reasonably accurate. Due to reflections from the tank ends and walls, it was not possible to run the experiments continuously for more than about 20 min. before the wave motion became unacceptably distorted. Thus difficulty was overcome by stopping the experiment and then restarting. Generally two 20 min. blocks were used for each target wave spectrum, containing a total of about 1200 consecutive rolls.

Four different wave target were chosen for the present study, as summarised in Table 1 (here converted to full scale).

Table 1

Dataset No.	Target Wave Spectrum		No of rolls recorded
	Type	Parameters	
1	JONSWAP	H=5·1 m,T=8·5 s	1001
2	JONSWAP	H=4·4 m,T=6·6 s	1325
3	ITTC	H=4·7 m	1249
4	JONSWAP	H=6·1 m,T=8·5 s	580

Fig. 5.5 shows a comparison between the target wave spectrum and spectral estimates from the actual wave record in the vicinity of the ship model, for dataset 1.

2.5.1 Conversion to roll moment spectra

The measured wave elevation spectra can be converted to wave moment spectra, by using equation (21).

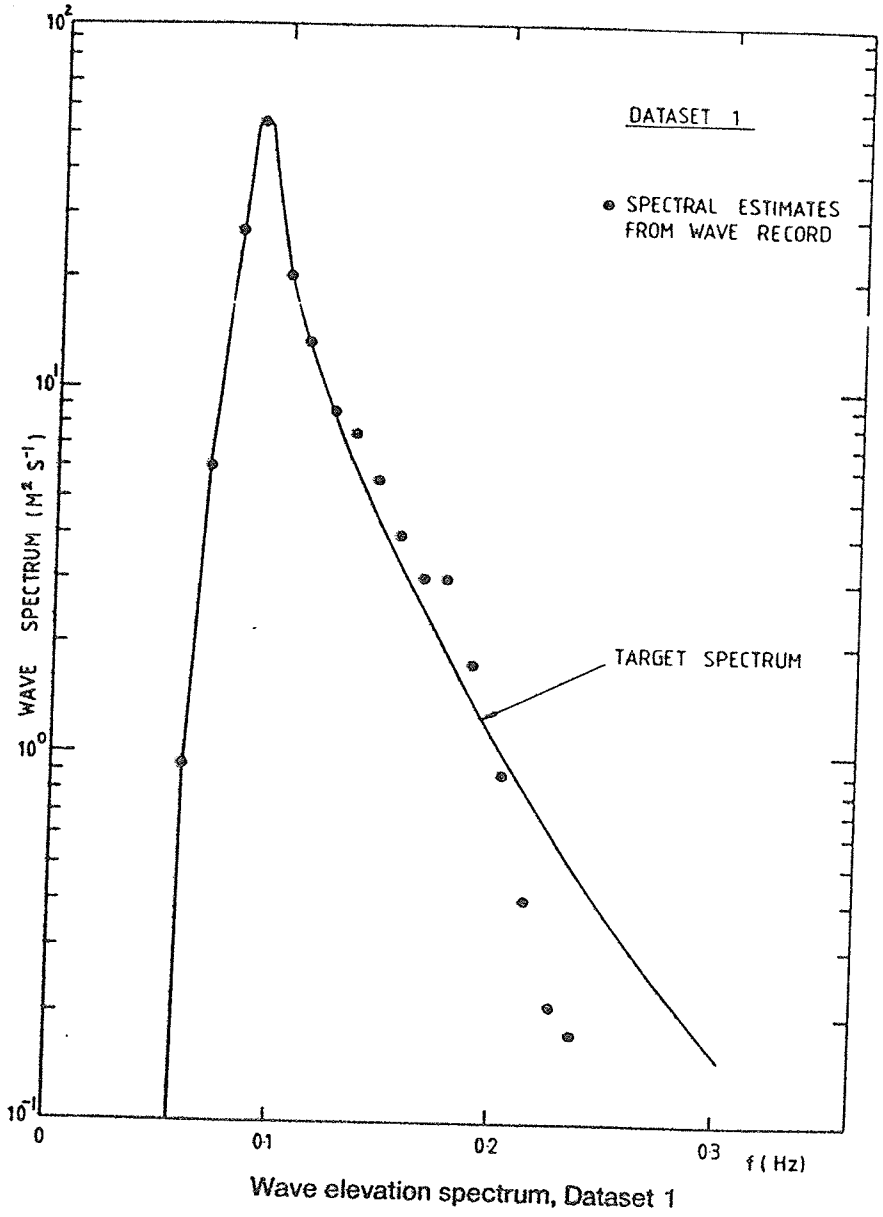


Fig. 5.5.

Fig. 5.6 shows a typical result of such a conversion (here Dataset 1). Here the broken line shows the result of converting the target wave spectrum and the full circles the result of converting the experimental estimates of wave spectra. Since $M(\omega)$ increases rapidly with frequency, the conversion process has the effect of considerably magnifying the "tail" of the wave elevation spectrum. The experimental results have significant scatter and for computational purposes the continuous curve shown

by a full line was used. Also shown in Fig. 5.6, for comparison purposes, is the roll displacement spectrum, $S_R(\omega)$, as estimated from the digitised roll response records. As expected this shows a sharp peak, close to $f = 0.113$ Hz , the theoretical natural roll frequency. It is interesting to note that a secondary peak appears in the roll displacement spectrum, at the frequency at which $S_M(\omega)$ peaks. This has the effect of broadening the bandwidth of the roll response spectrum.

A basic assumption of the theory is that $S_R(\omega)$ has a band-width which is appreciably less than the band-width of $S_M(\omega)$. Fig. 5.6 shows that this assumption is not unreasonable in the present study.

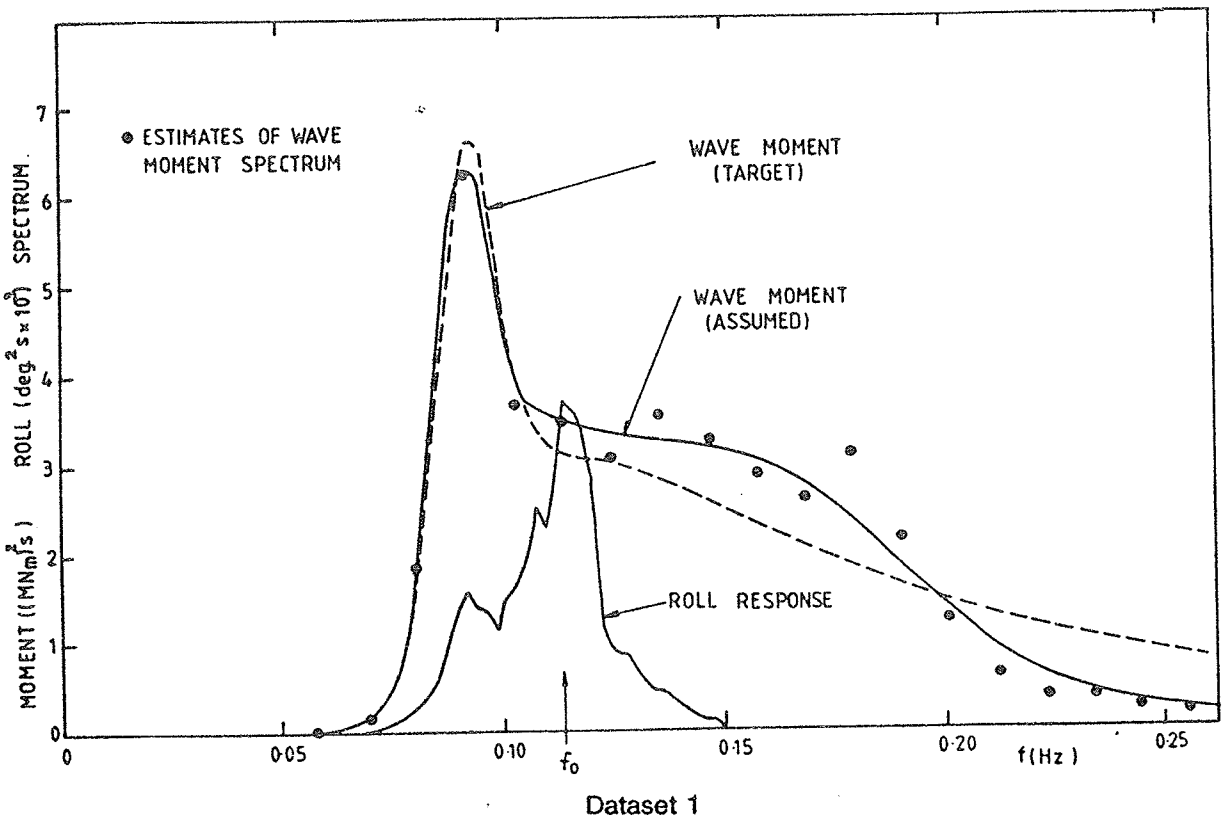


Fig. 5.6.

2.5.2 Standard deviation of roll response

From the theoretical analysis described earlier, the roll standard deviation, σ , can be calculated, using the original and modified (i.e. with spectral level correction factor) theories. Table 2 gives a comparison between experimental and theoretical values of σ .

Table 2

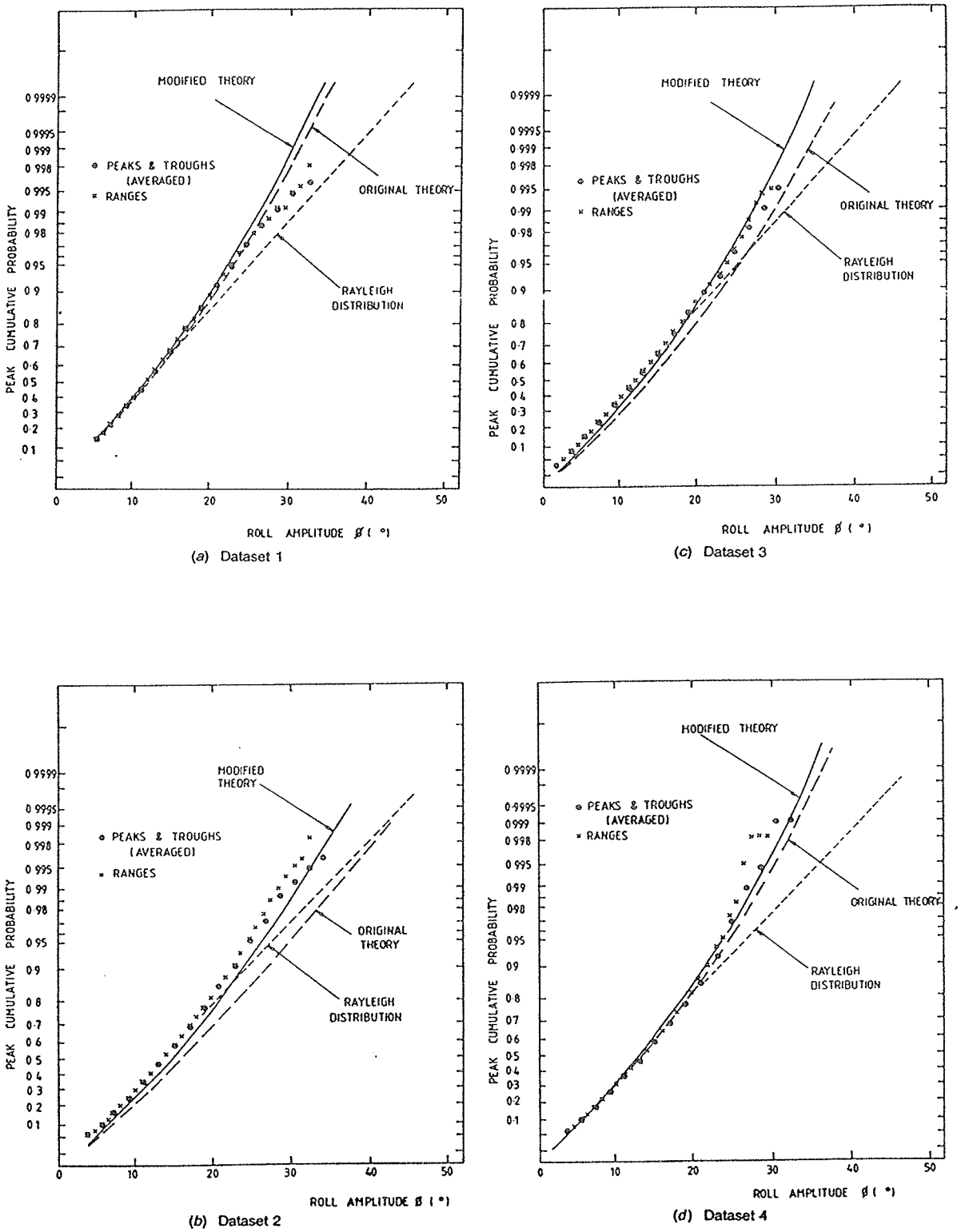
Dataset	Experimental σ deg.	Theoretical σ deg.	
		original	modified
1	10.1	10.1	9.9
2	11.3	12.9	11.8
3	10.4	11.1	10.4
4	11.4	10.8	10.7

2.5.3 Cumulative probabilities of roll peak amplitudes

The cumulative distribution, P(A) of the roll peak amplitudes is given by

$$P(A) = \int_0^A w(a)da \qquad (39)$$

Figs. 5.7(a) to (d) show, for Datasets 1 to 4, respectively, the variation of P(A) with roll amplitude, as obtained from the theory (original and modified) and from the experimental roll records. Also shown, for comparison purposes, is the Rayleigh distribution for each case (based on the experimental value of the standard deviation). Rayleigh probability paper has been used in the presentation of these results; thus Rayleigh distributions appear as straight lines.



Roll peak cumulative probability—comparison between theory and experiment

Fig. 5.7.

To obtain the experimental estimates the positive peak amplitude, a^+ , measured from zero level, were first used to generate a histogram. A similar histogram was also formed from the negative peak (or trough) amplitudes a^- . The average of these two histograms was then found. A histogram of the peak to trough range values, r , was also found and converted into a histogram of half-range values, $r/2$. These histograms were then used to construct estimates of $P(A)$.

It is observed that, in all four cases, the modified theory gives fairly good agreement with the experimental results; the pronounced deviation from the Rayleigh distribution, at high roll amplitudes, is well predicted by the theory. This deviation is principally due to the strong non-linearity in the damping, which is almost entirely quadratic. At high amplitudes the Rayleigh distribution seriously overestimates the probability of reaching these levels of response.

3. Hysteretic oscillator response to wide-band random excitation

For the purpose of predicting the response of hysteretic oscillators to random excitation, it was shown some time ago, by the present author (Roberts (1978),(1980)) that the method of stochastic averaging offers a useful alternative approach to equivalent linearisation, in situations where the energy dissipation per cycle is relatively small. In the specific case of an oscillator with a bilinear hysteretic force, a modified form of the standard stochastic averaging method was used to obtain analytical expressions for the response distribution (Roberts (1978)). For cases where the response is narrow-band in nature, this theory gives results in very good agreement with digital simulation estimates. Later this theoretical technique was used to give useful information concerning the yield statistics of a simple elasto-plastic oscillator (Roberts (1980)).

Here (see also Roberts (1988)) it will be shown that the stochastic averaging method can also be applied in situations where the hysteresis loop is modelled through an extended differential equation, of the kind proposed by Bouc (1967), and used extensively by many authors, in connection with the equivalent linearisation method.

3.1 The extended differential model

An oscillator with the following differential equation of motion will be considered:

$$\ddot{x} + 2\xi\omega_0\dot{x} + \alpha\omega_0^2 x + (1-\alpha)\omega_0^2 z = f(t) \qquad (40)$$

Here x is a non-dimensional displacement (normalised by a characteristic yield displacement), ζ is a non-dimensional damping ratio, ω_0 is the "pre-yielding" natural frequency, α is the post to preyielding stiffness ratio, $f(t)$ is the excitation and z is a non-dimensional hysteretic restoring force z can be conveniently modelled in terms of the differential equation

$$\dot{z} = -\gamma|\dot{x}|z|z|^{n-1} - \beta\dot{x}|z|^n + A\dot{x} \qquad (41)$$

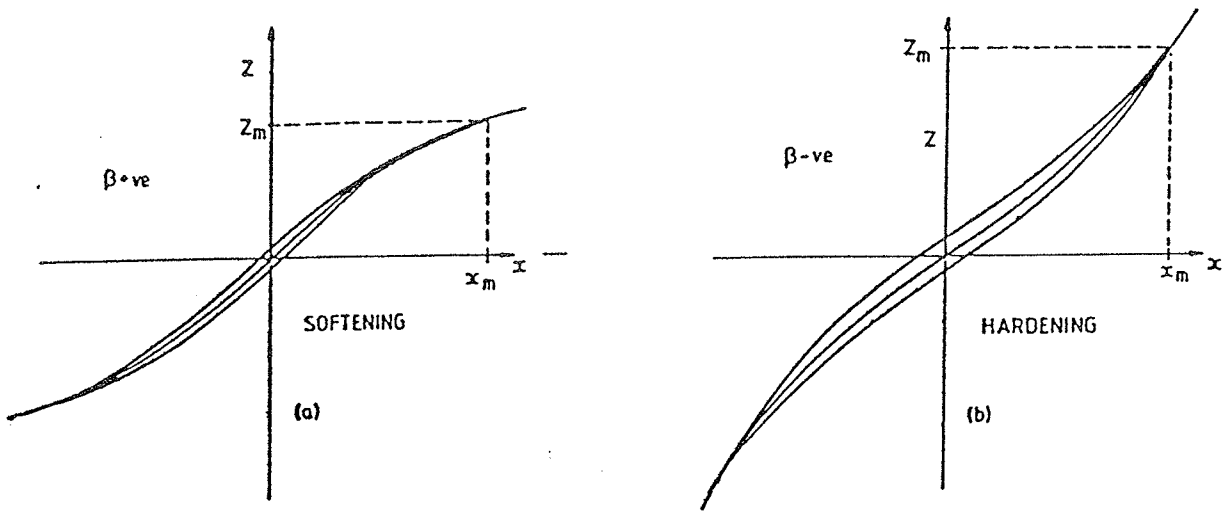


Fig. 5.8.

(see Chapter One). Here the parameters n, γ, β and A control the magnitude and shape of the hysteresis loop.

Here attention is focussed on cases where the energy lost per cycle is relatively small - i.e. the hysteresis loop is "slim" and the area enclosed is relatively small. Referring to Fig. 5.8 it is noted that such loops can be considered to have a non-linear "backbone", g(x) , which may be of the stiffening or softening type, depending on the sign of β . One can write

$$z(x) = g(x) + \epsilon(x) \tag{42}$$

where $\epsilon(x)$ is the purely hysteretic component of z(t) .

Equation (42) represents, roughly speaking, a decomposition of z(x) into a non-linear stiffness effect, g(x) , and a non-linear damping effect. $\epsilon(x)$. Substituting from equation (42) into equation (40) one obtains

$$\ddot{x} + 2\zeta\omega_0\dot{x} + G(x) + \omega_0^2(1-\alpha)\epsilon(x) = f(t) \tag{43}$$

where

$$G(x) = \omega_0^2[(1-\alpha)g(x) + \alpha x] \tag{44}$$

3.2 Application of the averaging method

When the energy dissipation is small, the total energy

$$E(t) = \frac{\dot{x}^2}{2} + V(x) \tag{45}$$

where here

$$V(x) = \int_0^x G(\xi)d\xi \tag{46}$$

is the potential energy function, will be slowly varying. From equation (43) one has

$$\dot{E} = -\dot{x}[2\zeta\omega_0\dot{x} + \omega_0^2(1-\alpha)\epsilon(x)] + \dot{x}f(t) \tag{47}$$

As we have seen earlier, the first term on the right side of this equation is the rate of energy dissipation. Averaging this over one cycle, one has

$$H(E) = \frac{1}{T(E)}\oint \dot{x}[2\zeta\omega\dot{x} + \omega_0^2(1-\alpha)\epsilon(x)]dt \tag{48}$$

where $T(E)$, as before, is the period of free, undamped oscillations.

Equation (47) is now approximated by

$$\dot{E} = -H(E) + \dot{x}f(t) \tag{49}$$

and, if the excitation is approximated as a white noise, one is led, as shown in Chapter 3, to a Markov model for $E(t)$, with the governing FPK equation

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial E}\left\{\left[H(E) - \frac{I}{2}\right]p\right\} + \frac{I}{2}\frac{\partial^2}{\partial E^2}\left[C(E)p\right] \tag{50}$$

Here I, $C(E)$ and $p(E|E_0;t)$ have the same meaning as before. Previous results, in this Chapter, and in Chapter Three may now be used to obtain the stationary probability density functions of E and the joint process $[x,\dot{x}]$.

In the following, for simplicity, only the special case where

$$\alpha = \zeta = 0 \tag{51}$$

will be considered. In this situation

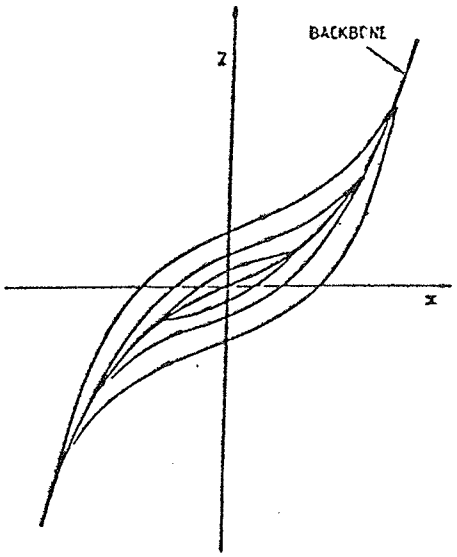


Fig. 5.9.

$$H(E) = \frac{A(E)}{T(E)} \tag{52}$$

where

$$A(E) = \omega_0^2 \oint \epsilon(x)dx = \omega_0^2 \int z\, dx \tag{53}$$

is the area enclosed by the hysteretic loop.

Also, in the following, attention will be devoted exclusively to the case where

$$n = 1 \quad , \quad \omega_0 = 1 \tag{54}$$

The condition $\omega_0 = 1$ does not imply any loss of generality, since changing the value of ω_0 is equivalent to rescaling the strength, I , of the input.

3.3 Calculation of the backbone

There are several ways of defining the "backbone", illustrated in Fig. 5.8. It is observed that, for fixed values of A, β and γ , there is a family of possible loops, as sketched in

Fig. 5.9. A convenient definition of the backbone, used here, is the locus of the extremities (x_m, z_m) of the loop. Thus $g(x)$ is defined by

$$g(x_m) = z_m \tag{55}$$

The relationship between x_m and z_m is easily obtained by integrating equation (41), and using a symmetry condition. One finds that

$$2x_m = -\frac{1}{\lambda_u} \ln\left[1 - \frac{\lambda_u z_m}{A}\right] - \frac{1}{\lambda_d} \ln\left[1 - \frac{\lambda_d z_m}{A}\right] \tag{56}$$

where

$$\left.\begin{aligned} \lambda_u &= \beta + \gamma \\ \lambda_d &= \beta - \gamma \end{aligned}\right\} \tag{57}$$

This non-linear algebraic relationship must, in general, be solved numerically. However, if γ is small the hysteresis loop is slim and one can use the following asymptotic approximation:

$$g(x) = \frac{A}{\beta}\,[1 - \exp(-\beta x)] \tag{58}$$

Irrespective of the value of γ, the backbone is linear as $x_m \to 0$; thus $g(x) \to Ax$.

3.4 Calculation of the area of the loop

An exact expression for the area, $A(E)$, of the loop can be easily obtained by using equation (53). The result is

$$A(E) = 2A\left\{ \left[\frac{\ln(1-\psi_0)+\psi_0}{\lambda_d^2}\right] - \left[\frac{\ln(1-\theta_0)+\theta_0}{\lambda_u^2}\right] \right\} \tag{59}$$

where

$$\psi_0 = \frac{\lambda_d z_m}{A} \quad , \quad \theta_0 = \frac{\lambda_u z_m}{A} \tag{60}$$

and E is related to z_m through the expressions

$$z_m = g(x_m) \quad ; \quad E = V(x_m) \tag{61}$$

Asymptotic expressions for $A(E)$ can be found in certain special cases. For example

$$A(E) \rightarrow \frac{8}{3}\left[\frac{2}{A}\right] \gamma E^{3/2} \qquad \text{as} \quad E \rightarrow 0 \tag{62}$$

irrespective of the magnitudes of β and γ. The above expressions for $A(E)$ also holds, asymptotically, for any E, if $\beta = 0$ and $\gamma \rightarrow 0$.

For $\beta > 0$ one finds that, as $E \rightarrow \infty$

$$A(E) \rightarrow \frac{2A}{\lambda_d} \ln\left[1 - \frac{\lambda_d}{\lambda_u}\right]\left[\frac{1}{\lambda_d} + \frac{1}{\lambda_u}\right] - \frac{2A}{\lambda_u}\left[\frac{1}{\lambda_u} - \frac{1}{\lambda_d}\right] + \frac{4Ax_m}{\lambda_u} \tag{63}$$

where $E = V(x_m)$. This expression is useful, for the purposes of numerical evaluation, since an evaluation of $A(E)$ using equation (59) can lead to numerical difficulties when E is large.

3.5 Calculation of $T(E)$, $C(E)$ and $D_2(E)$

Considering the evaluation of $T(E)$, the periodic time, initially, it is noted that the integrand in the expression

$$T(E) = 2\sqrt{2} \int_0^b \frac{dx}{\sqrt{E-V(x)}} \tag{64}$$

becomes infinite as $x \rightarrow b$. To remove this singularity it is convenient to use the transformation

$$V(x) = E \sin^2\theta \tag{65}$$

In terms of θ, equation (64) can be expressed as

$$T(E) = 3\sqrt{2E} \int_0^{\pi/2} \frac{\sin\theta}{g(x)}\, d\theta \tag{66}$$

As $\theta \to 0$, in the above integral, $x \to 0$, $V(x) \to Ax^2/2$ and hence the integrand approaches the finite value $1/(2VA)^{\frac{1}{2}}$. Thus the integral in equation (64) can be evaluated by straightforward methods.

Similarly, the functions $C(E)$ and $D_2(E)$ can be evaluated by a transformation to the θ variable, defined above. Thus

$$C(E) = \frac{8\sqrt{2}\ E^{3/2}}{T(E)} \int_0^{\pi/2} \frac{\cos^2\theta\sin\theta}{g(x)}\, d\theta \tag{67}$$

and

$$D_2(E) = \frac{4\sqrt{2E}}{T(E)} \int_0^{\pi/2} \frac{x^2\sin\theta}{g(x)}\, d\theta \tag{68}$$

Asymptotic expressions for the above functions of E are useful, for the purpose of evaluating $p(E)$ and σ^2. As $E \to 0$ the restoring function $g(x) \to Ax$, as previously mentioned. Hence one finds that

$$\left.\begin{aligned} T(E) &\to 2\pi A^{-\frac{1}{2}} \\ C(E) &\to E \\ D_2(E) &\to E/A \end{aligned}\right\} \quad \text{as} \quad E \to 0 \tag{69}$$

For $\beta > 0$ the following asymptotic expressions for the case $E \to \infty$ can also be found

$$\left.\begin{aligned} T(E) &\rightarrow 4\sqrt{2E}\ \beta/A \\ C(E) &\rightarrow 2E/3 \\ D_2(E) &\rightarrow \frac{8}{15}\left[\frac{E\beta}{A}\right]^2 \end{aligned}\right\} \qquad (70)$$

3.6 The loss factor

The function $H(E)$ may be related to the non-dimensional damping function

$$Q(E) = \frac{H(E)}{2\omega_0 E} \qquad (71)$$

(see Chapter 1, where it is pointed out that, for the linear oscillator with viscous damping, $Q(E) = \zeta$, where ζ is the usual critical damping factor).

Previous studies indicate that, in the linear case, the stochastic averaging approximation gives very good results if $\zeta < 0.1$. Accordingly, one can tentatively assume that, in the present application, the stochastic averaging approximation is likely to be reasonably accurate if $Q(E) < 0.1$.

Using the result given earlier in this paper it is easy to evaluate $Q(E)$ for a particular set of loop parameters (A, β and γ). Figs. 5.10(a) and (b) show typical variations of $Q(E)$ with E, for $A = 1$, and various β values. In Fig. 5.10(a) $\gamma = 0.5$ and in Fig. 5.10(b), $\gamma = 0.1$. Such results give an indication of the likely accuracy of the stochastic averaging method, for a particular loop parameter set, and a given E range.

3.7 The case $\beta = 0$

In the special case where $\beta = 0$, the backbone is linear. If, furthermore, γ is small, so that the asymptotic area expression given by equation (62) is relevant, then it is possible to

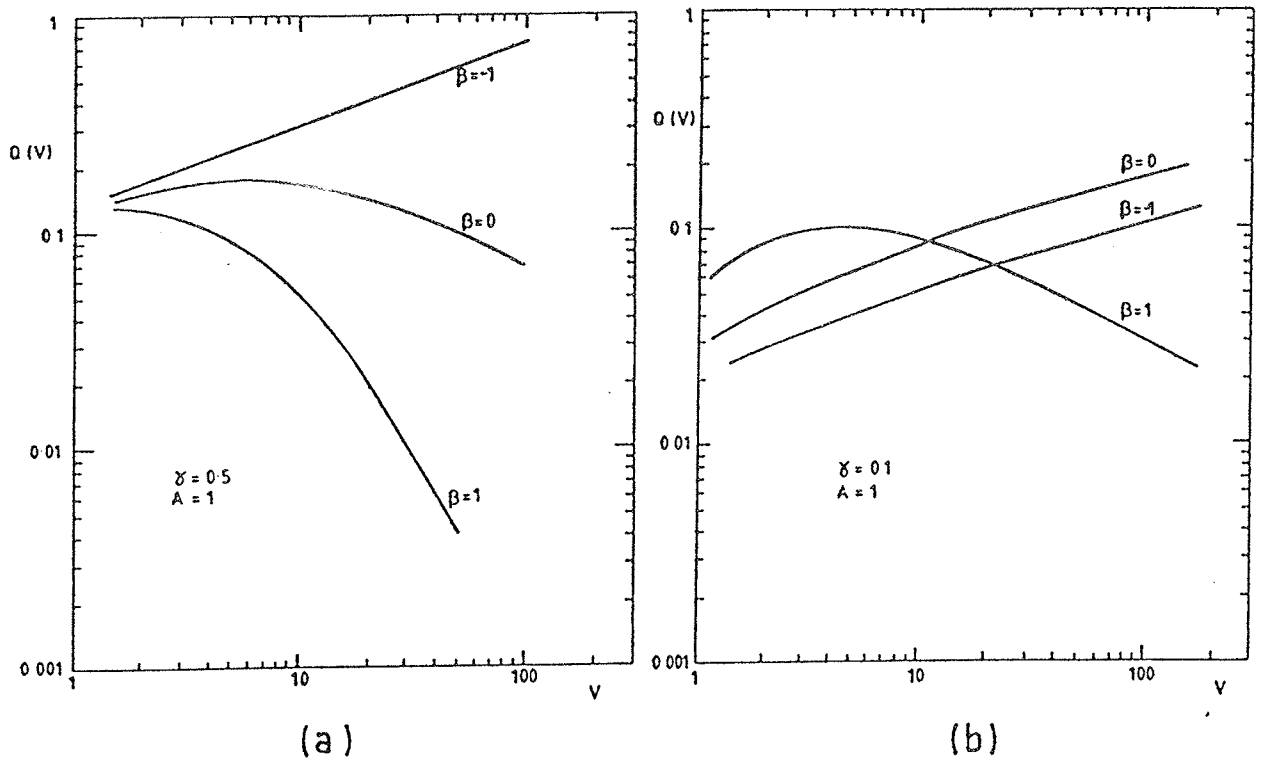


Fig. 5.10.

obtain a complete closed-form solution for $w(E)$, the stationary density function of E . Thus

$$\frac{H(E)}{C(E)} = \frac{8\sqrt{2}}{6\pi} E^{\frac{1}{2}} \gamma \qquad (\gamma \to 0) \qquad (72)$$

and hence

$$p(E) = k \exp\left\{- \frac{\rho\gamma}{I} V^{3/2}\right\} \qquad (\gamma \to 0) \qquad (73)$$

where

$$\rho = \frac{16\sqrt{2}}{9\pi} \sim 0.800 \qquad (74)$$

and

$$k = \frac{3}{2}\left[\frac{\rho\gamma}{I}\right]^{\frac{2}{3}} \frac{1}{\Gamma(2/3)} \qquad (75)$$

Here $\Gamma(\ )$ is the Gamma function.

An explicit expression for the mean square displacement, σ^2 , can also be found. The result is

$$\sigma^2 = \frac{\lambda}{A}\left[\frac{I}{\rho\gamma}\right]^{2/3} \qquad (\text{as} \quad \gamma \to 0) \qquad (76)$$

where

$$\lambda = \frac{\Gamma(1/3)}{3\Gamma(2/3)} \sim 0.66 \qquad (77)$$

3.8 Comparisons with simulation results

Simulation estimates of σ and $p(E)$ where obtained by numerically solving the differential equations of motion, using the fourth order Runge-Kulta algorithm.

A useful, preliminary assessment of the accuracy and range of validity of the present theory can be obtained by comparing theoretical estimates of the standard deviation of the response, σ , with corresponding simulation estimates.

Figs. 5.11 and 5.12 show a set of such comparisons for the case where $\gamma = 0.02$ and 0.1 , respectively and β ranges from 0 to -5 . Here the ratio σ/D is plotted versus D , where $D = (I/\pi)^{1/2}$ is an input level parameter. It is observed that, for negative β (a hardening $g(x)$ function) the present theory gives excellent agreement with the simulation estimates. Very good agreement is also obtained for the case $\beta = 0$, if D is relatively small. However, for high values of D the simulation estimates tend to fall above the theoretical line.

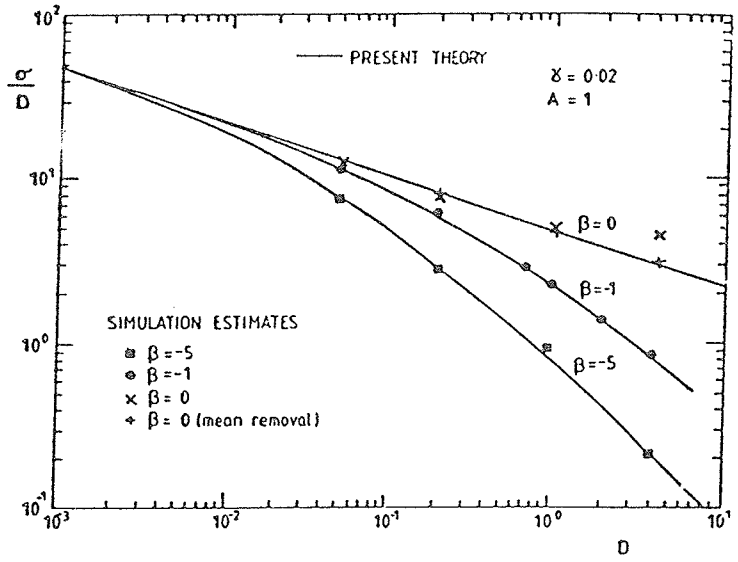


Fig. 5.11.

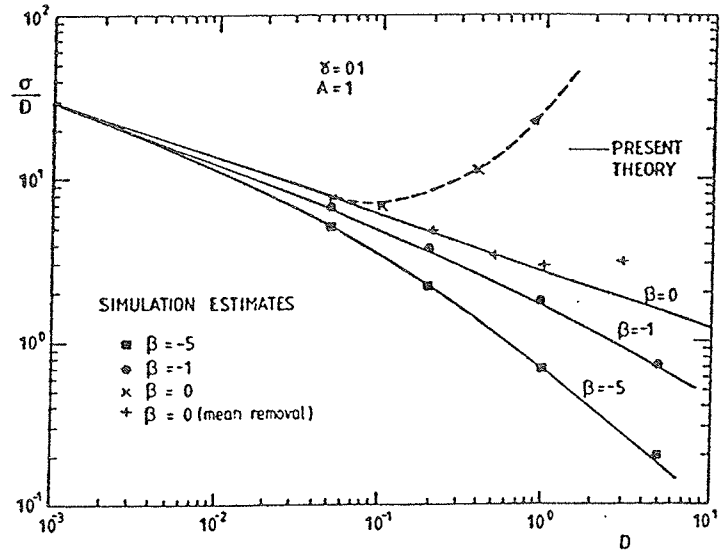


Fig. 5.12.

The latter discrepancy is due to the well-known drifting phenomenon, whereby the "centre" of the oscillation wanders very slowly. Drifting becomes noticeable if, as the extremities of the hysteresis loops are approached, the stiffness dz/dx approaches zero. For the class of oscillators under consideration, it is easy to show that $dz/dx \to 0$ as $z_m \to \infty$, provided that $\lambda_u = \beta+\gamma$ is positive. As D increases the probability of high z_m values being reached increases and hence the tendency to drift increases.

One can represent $x(t)$ as

$$x(t) = y(t) + d(t) \tag{78}$$

where $d(t)$ is the very low frequency drift motion and $y(t)$ is the oscillatory response, the periods of which are related to $T(E)$. The present theory does not account for drift motion but does relate to the process $y(t)$.

The non-drifting motion can be estimated from the digital simulation values of $x(t_i)$ $(t_i = i\Delta t)$ by using a high pass digital filtering operation to remove $d(t)$. A simple form of this, used here, is

$$y(t_i) = x(t_i) - \frac{1}{n} \sum_{k=0}^{m-1} x(t_{i-k}) \tag{79}$$

The second term on the right hand side of this equation represents a "local mean" of $x(t_i)$. If $m\Delta t > T^*$, where T^* is a characteristic period of oscillation of $y(t_i)$, then one can expect to remove virtually all of $d(t)$ from the response samples.

Values of σ estimated from the modified data, $y(t_i)$, are shown by crosses (+) in Figs. 5.11 and 5.12. At the highest value of D , these estimates are seen to be in good agreement with the present theory, confirming that the latter is a good model of the non-drifting component, $y(t)$, of the response.

Finally, to demonstrate the fact that the present theory enables the distribution of the response to be estimated, several comparisons between theoretical estimates of the density function, $p(E)$ and the corresponding simulation estimates are shown in Figs. 5.13(a),(b) and (c). In Fig. 5.13(a) and (b) the simulation estimates were derived from the original data, $x(t_i)$ (i.e., no drift removal). As the standard deviation results would lead

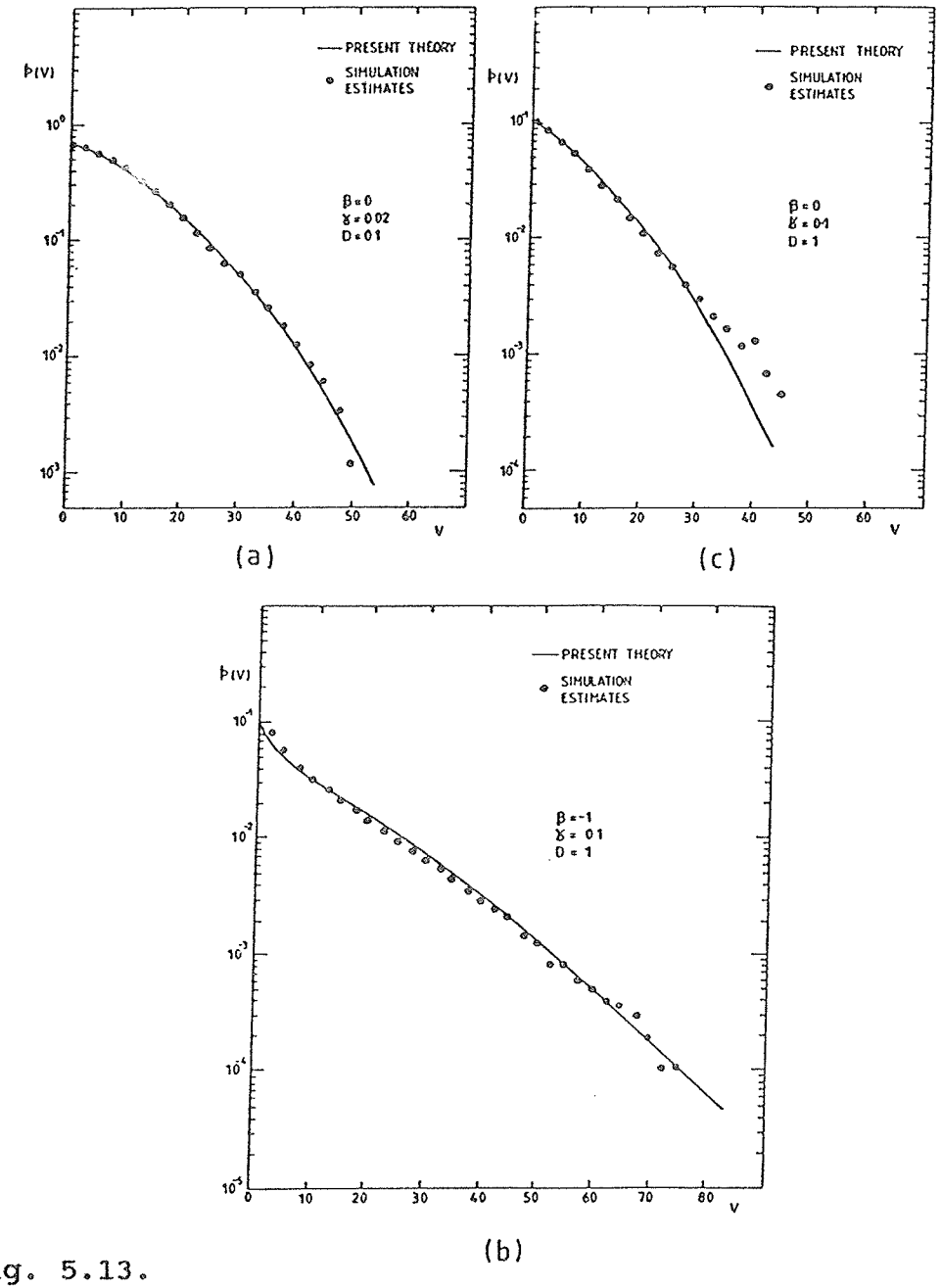


Fig. 5.13.

one to expect, the theory agrees very well with the simulation estimates, in both cases. In Fig. 5.13(c) some results are shown for a case where significant drifting occurs; here the simulation estimates are derived from the modified data, $y(t_i)$. Again the agreement is satisfactory but there is a tendency for the simulation estimates to fall above the theoretical line, at high energy levels, indicating that the effect of drifting is still present, to some extent, in the modified data.

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