Afdelingen for Bærende Konstruktioner
Department of Structural Engineering
Danmarks Tekniske Universitet • Technical University of Denmark

STRENGTH OF CRACKED CONCRETE

Part 1 --- Shear Strength of Conventional Reinforced Concrete Beams, Deep Beams, Corbels, and Prestressed Reinforced Concrete Beams without Shear Reinforcement

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PREFACE

This paper has been prepared as a part of the thesis required to obtain the Ph.D. degree at The Technical University of Denmark.

The work has been carried out at the Department of Structural Engineering under the supervision of Professor, dr.techn. M.P.Nielsen.

I wish to express my sincere appreciation to my supervisor for his inspiring advises and encouragement, and to the entire staff at the department for their help during the time I have been here.

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Lyngby, 1994

Jin-Ping Zhang

SUMMARY

This report deals with the shear strength of reinforced concrete beams without shear reinforcement determined by means of the theory of plasticity.

It has been observed that in general the shear failure of non shear reinforced concrete beams is featured by the formation of a critical diagonal crack. In this paper, a physical explanation is given for this fact under the hypothesis that the cracking of concrete introduces potential yield lines which may be more dangerous than the ones found by the usual plastic theory. The shear strength of cracked concrete is expressed by introducing a reduction factor which reflects the reduction of sliding strength due to cracking.

Conventional reinforced concrete beams, subjected to both concentrated loading and uniform loading, deep beams and corbels, and prestressed beams are studied. The theoretical calculations are compared with experimental results reported in the literature.

A good agreement has been found.

RESUMÉ

Denne rapport behandler bæreevnen af armerede betonbjælker og betonplader uden forskydningsarmering bestemt v.h.a. plasticitetsteorien.

Forskydningsbrud i ikke forskydningsarmerede bjælker og plader er karakteriseret ved, at der dannes en såkaldt kritisk diagonal revne. I rapporten gives en fysisk forklaring på dette fænomen, der går ud på, at revnedannelse i betonen introducerer potentielle brudlinier eller flydelinier, som kan være farligere end dem, der findes v.h.a. sædvanlig plasticitetsteori. Glidningsmodstanden i en revne beregnes ud fra glidningsmodstanden i urevnet beton v.h.a. en reduktionsfaktor.

Både slapt armerede bjælker påvirket af koncentreret last og jævnt fordelt last, høje bjælker, konsoller og forspændte bjælker og plader studeres. De teoretiske resultater sammenlignes med forsøgsresultater fra litteraturen.

Der er fundet god overensstemmelse.

TABLE OF CONTENTS

NOTATIONS

a	: Length of shear span
a_0	: Horizontal distance between the end of the yield line and the support in uniformly loaded beams
a/h	: Shear span ratio
A_s	: Cross sectional area of longitudinal reinforcement
b	: Web width of beam
c	: Cohesion of uncracked (micro-cracked) concrete
c'	: Cohesion of cracked (macro-cracked) concrete
d	: Diameter of longitudinal reinforcement, effective depth of beam
f_1	: Function of compressive strength of concrete
f_2	: Function of depth of beam
f_3	: Function of reinforcement ratio
f_4	: Function of prestressing level
f_c	: Uniaxial compressive strength of concrete
f_{c0}^*	: Effective compressive strength of uncracked (micro-cracked) concrete, defined as $f_{c0}^*=\nu_0 f_c$
f_c^*	: Effective compressive strength of cracked (macro-cracked) concrete, defined as $f_c^*=\nu f_c$
F_{se}	: Effective prestressing force in the longitudinal reinforcement
f_t	: Tensile strength of concrete
f_t^*	: Plastic equivalent tensile strength of concrete
f_Y	: Yield strength of longitudinal reinforcement
h	: Depth of beam
h_e	: Effective depth of beam
h_i	: Lever arm, i.e. the distance between the tensile and the compression resultant

k	: Material constant, $k=(1+\sin\varphi)/(1-\sin\varphi)$
L_0	: Length of load and support platen
L	: Clear span of beam
L_{cr}	: Mean distance between cracks
$L_{max.}$	: Maximum distance between cracks
M_{cr}	: Cracking moment
p	: External load per unit area
P	: External load
P_{cr}	: Diagonal cracking load
P_u	: Ultimate load or load carrying capacity
$s(h)$	: Size effect parameter
u	: Relative displacement in yield line
V	: Reaction force at support
V'	: Shear force at the edge of support platen
W_I	: Internal work
W_E	: External work
x	: Starting position of critical diagonal crack, i.e. horizontal distance from the bottom of a critical diagonal crack to the edge of support platen
y_0	: Depth of compression zone
α	: Inclination of relative displacement with respect to yield line
β	: Inclination of yield line
γ	: Plasticity factor of resistance
θ	: Inclination of diagonal crack
μ	: Coefficient of friction for uncracked (micro-cracked) concrete, $\mu=\tan\varphi$
μ'	: Coefficient of friction for cracked (macro-cracked) concrete,

$\mu' = \tan\varphi'$

ν_0	: Effectiveness factor for uncracked (micro-cracked) concrete
ν_s	: Sliding reduction factor due to cracking
ν	: Effectiveness factor for cracked (macro-cracked) concrete
ρ	: Reinforcement ratio
σ	: Normal stress
$\sigma_1, \sigma_2, \sigma_3$	: Principal stresses
σ_{se}	: Average prestressing stress over whole section, $\sigma_{se} = F_{se}/bh$
σ_{se}/f_c	: Level of prestressing
τ	: Shear stress
τ_0	: Shear strength of uncracked (micro-cracked) concrete
τ_u	: Shear strength of cracked (macro-cracked) concrete
τ_{cr}	: Shear strength determined by diagonal cracking load
φ	: Angle of friction for uncracked (micro-cracked) concrete
φ'	: Angle of friction for cracked (macro-cracked) concrete
Φ	: Mechanical degree of longitudinal reinforcement, $\Phi = A_s f_Y / bhf_c$

I. INTRODUCTION

The problem of shear behaviour and shear strength of reinforced concrete beams and slabs has been perplexing researchers and engineers for many years. Numerous tests have been carried out to investigate the shear behaviour, and various models and methods have been suggested to describe the shear failure and to predict the shear strength [67,2][93,1]. Great progress has been achieved both in theory and applications, however it is a fact that fundamentally a comprehensive shear theory still has to be looked for.

Among the many attempts, the plastic theory for non shear reinforced concrete beams and slabs was developed in the 70's by works of Nielsen et al [78,1], B.C.Jensen[77,2], M.W.Bræstrup[78,1][78,2], M. Roikjær [77,1], J.F.Jensen [78,2][81,1], B.Feddersen [83,1], G.W.Chen [88,1] and others.

It turned out that in order to get good agreement between theory and experiments, it was necessary to introduce an effectiveness factor ν which, besides the usual and easily understandable parameters, contained the shear span ratio. The function, which was suggested by Roikjær [77,1], gave high values of ν for short shear spans, and the value of ν decreased when the shear span ratio approached a value about 2.6, and then increased again for increasing shear span ratio.

The physical meaning of this function was never really understood. Several attempts were made to explain it. Dowel action and the effect of the tensile strength of the concrete were considered, but no satisfactory explanation was obtained.

For a long time, it has been noticed that in general the shear failure of concrete beams is featured by the formation of a critical diagonal crack. In this paper, a physical explanation is given to account for this phenomenon. The main content of this explanation is that the cracking of concrete introduces potential yield lines which might be more dangerous than the ones found by the usual plastic theory. This is due to the fact that cracking reduces the sliding strength of the concrete to a value considerably lower than that for concrete without cracks. Measurements showing this fact have been done by tests with the so called push off or pull off tests. These tests reveal that the cohesion in a crack is only about half or less of the value for uncracked concrete.

Using this hypothesis it is shown in the paper that the load carrying capacity of reinforced concrete beams may be calculated using plastic theory with an effectiveness factor containing only quite natural parameters such as the uniaxial compressive strength of concrete, the reinforcement ratio, a size effect parameter, for instance the depth of the beam, etc. Further reduction on shear strength due to cracking is taken into account as well.

The hypothesis described in this paper in fact leads to a theory covering both conventional reinforced concrete beams and prestressed reinforced concrete beams. It also explains the difference of load carrying capacity between the beams with bond and those without bond.

Compared to the existing empirical methods, this theory offers considerable advantages. Firstly, the shear failure mechanism in the theory leads to a thorough understanding of the shear failure behaviour of the beams. Secondly, the theory unifies shear analysis of conventional reinforced concrete beams, deep beams and corbels, and prestressed beams, and the principles of the theory may be applied to any loading condition. Thirdly, by means of this theory, the ultimate shear capacity of cracked reinforced concrete beams can be calculated effectively, and even the position of the critical diagonal crack may be predicted approximately.

The theoretical calculations are compared with a wide range of available test results found in literature. A satisfactorily good agreement has been found.

II. SHEAR FAILURE MECHANISM OF REINFORCED CONCRETE BEAMS WITH CRITICAL DIAGONAL CRACKS

2.1 Diagonal cracks and shear failure

Tests on simply supported reinforced concrete beams without shear reinforcement under concentrated loads done all over the world have been reported in numerous papers. The general shear failure phenomenon can be summarized as following.

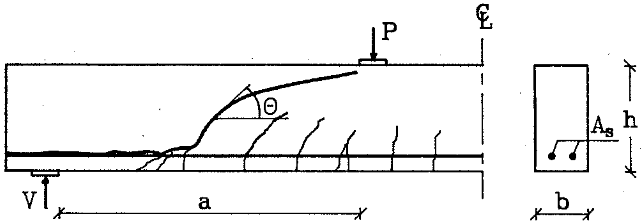


Fig.2.1.1 Crack pattern in specimen under symmetrical concentrated loading

For a beam with rectangular cross section subjected to two symmetrical concentrated loads (as shown in Fig.2.1.1, where the shear span is denoted as a), it is observed that the first group of flexural cracks occur in the shear stress free part, where the bending moment is constant. The cracks in this group are nearly perpendicular to the axis of the beam, and stop around the neutral axis. With the increase of loading, cracks appear in the shear span. At the beginning, the cracks are almost vertical, obviously initiating from bending. Above the level of reinforcement, due to the presence of shear stresses in this area, the cracks bend towards the axis of the beam, forming an angle θ to the axis which is less than 90°. The angle θ is changing along the path of crack, the longer the crack proceeds, the smaller the angle becomes. Thus diagonal cracks are formed.

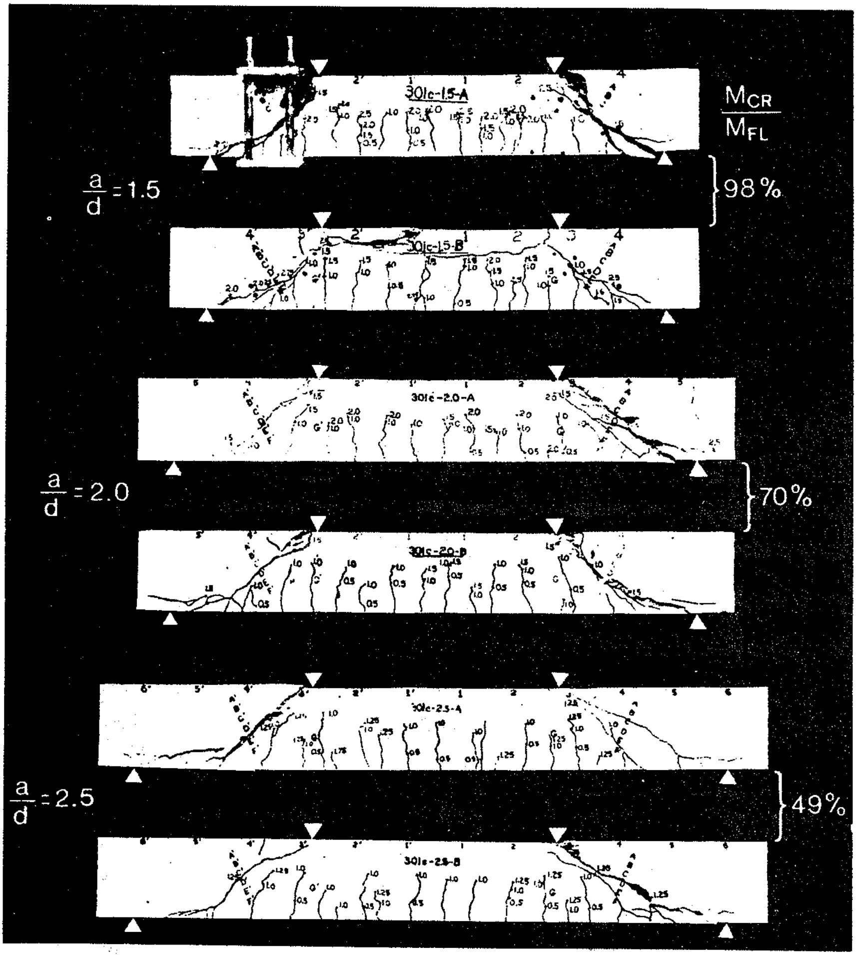
These cracks normally stop at a higher level than the pure bending cracks. Then as the load goes up, more and more diagonal cracks are formed. This development of diagonal cracks stops at a certain load level when the last diagonal crack suddenly propagates through nearly the total depth of the beam, and at the same time or prior to this, splitting between longitudinal steel bars and surrounding concrete around the area of the supports happens. The specimen either collapses simultaneously with the appearance of this diagonal crack or continues to sustain higher load until the concrete in the compression zone is crushed.

Fig.2.1.2 illustrates typical crack patterns of beams failed in shear in the test series by Kani[64,1][67,1], Leonhardt[62,1], Taylor,H.P.J.[68,1] and Taylor,R.[60,1][63,2].

It is observed in experiments that when the shear span ratio a/h is within a certain range, the shear failure of beams without shear reinforcement is always accompanied by diagonal cracks, and the upper tip of the last diagonal crack tends to end in the compression zone under the loading platen or at the edge of the loading platen, whereas the final failure line tends to end at the edge of the load platen. Due to the stress redistribution, other secondary cracks may appear.

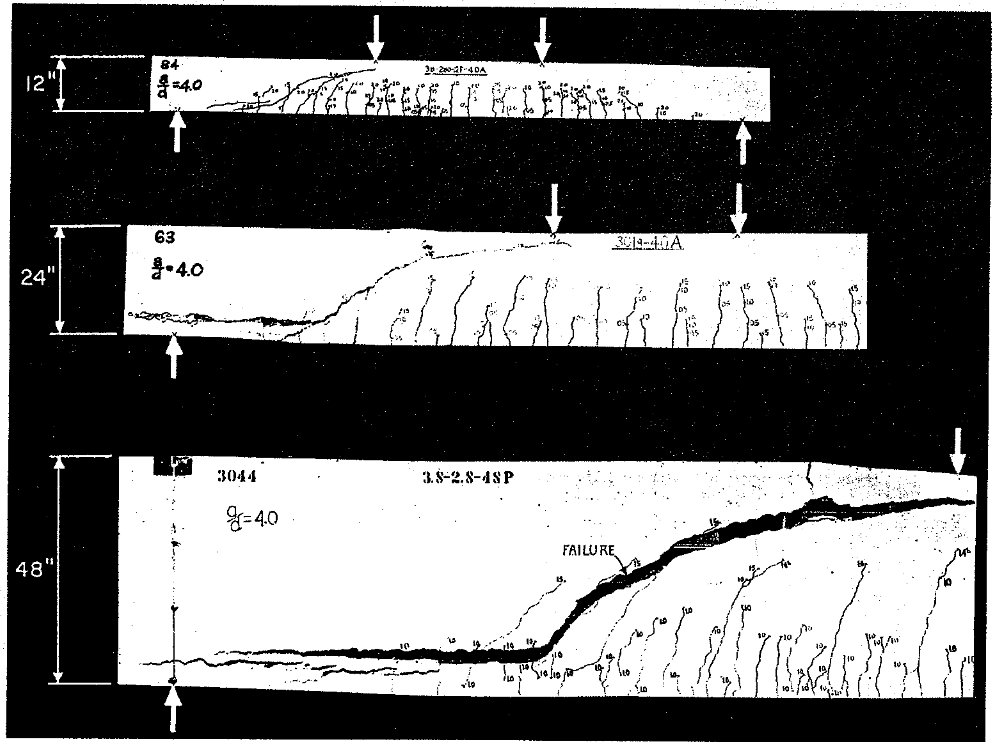
From these pictures of crack patterns of the specimens, it is clear that after the formation of the last diagonal crack, caused by relative displacements perpendicular to the crack, sliding occurs along the diagonal crack and the two parts on both sides of the crack shear off at failure.

In this paper, this last diagonal crack is referred to as the critical diagonal crack



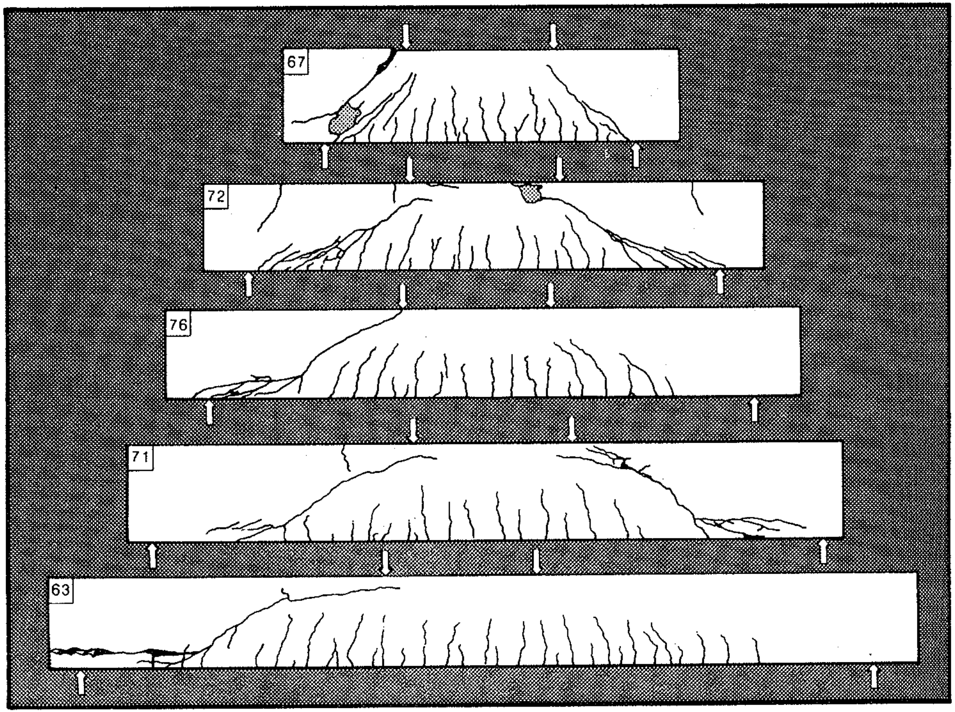
a). Typical failure patterns of beams tested by Kani [64,1]

Fig.2.1.2 Crack patterns of non shear reinforced concrete beams at shear failure



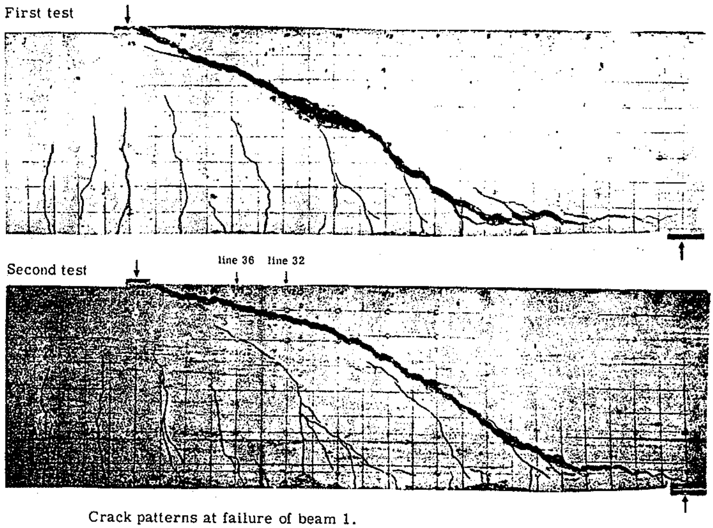
b). Crack patterns of beams with different depths tested by Kani [67,1]

Fig.2.1.2 (continued)

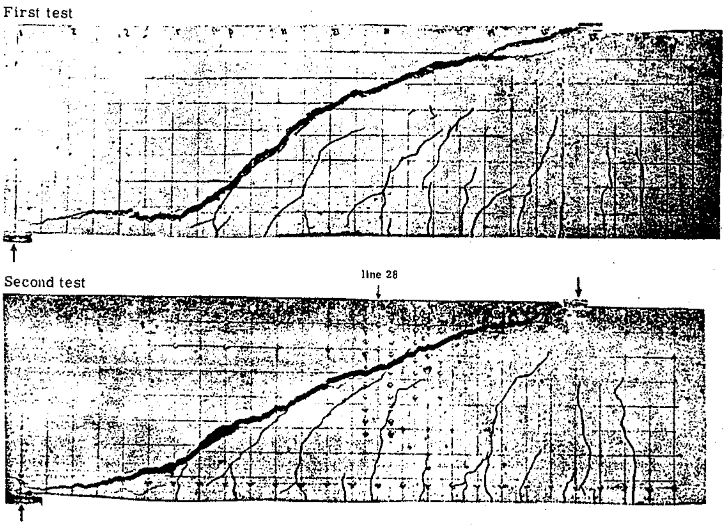


c). Crack patterns of beams at failure tested by Kani [91,1]

Fig.2.1.2 (continued)

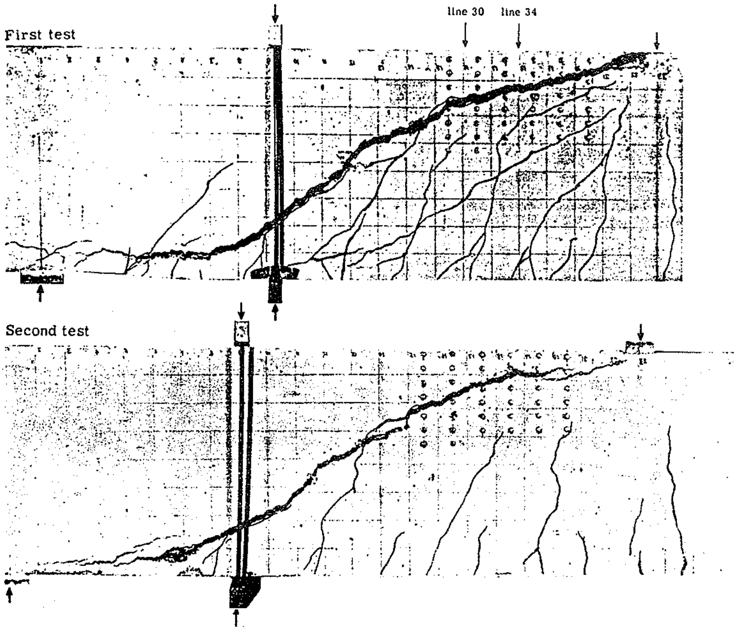


Crack patterns at failure of beam 1.

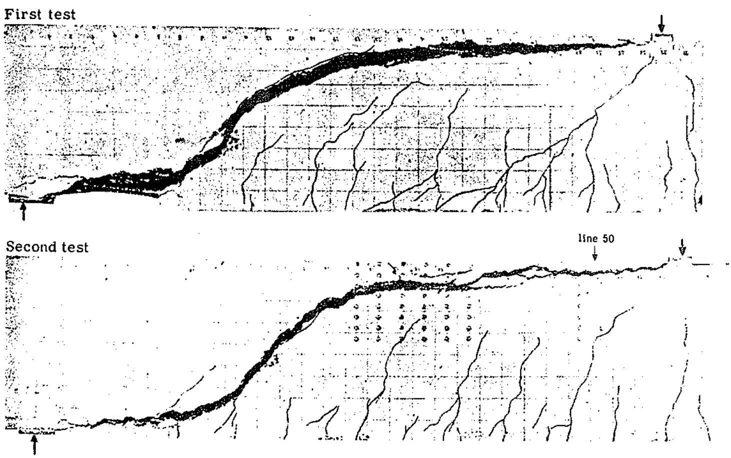


Crack patterns at failure of beam 2.

d). Crack patterns at failure of beams tested by Taylor, H.P.J. [68,1]
Fig.2.1.2 (continued).



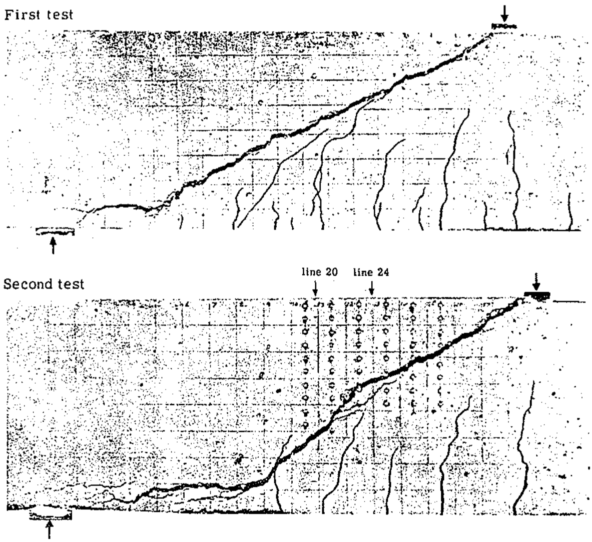
Crack patterns at failure of beam 3.



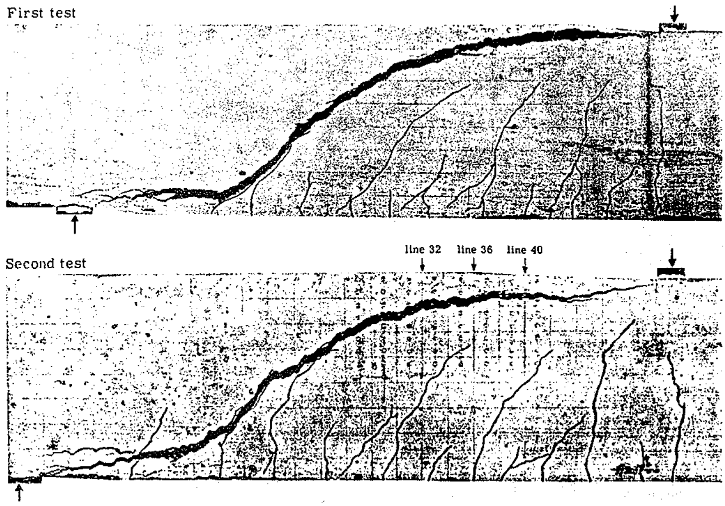
Crack patterns at failure of beam 4.

d). (continued).

Fig.2.1.2 (continued)



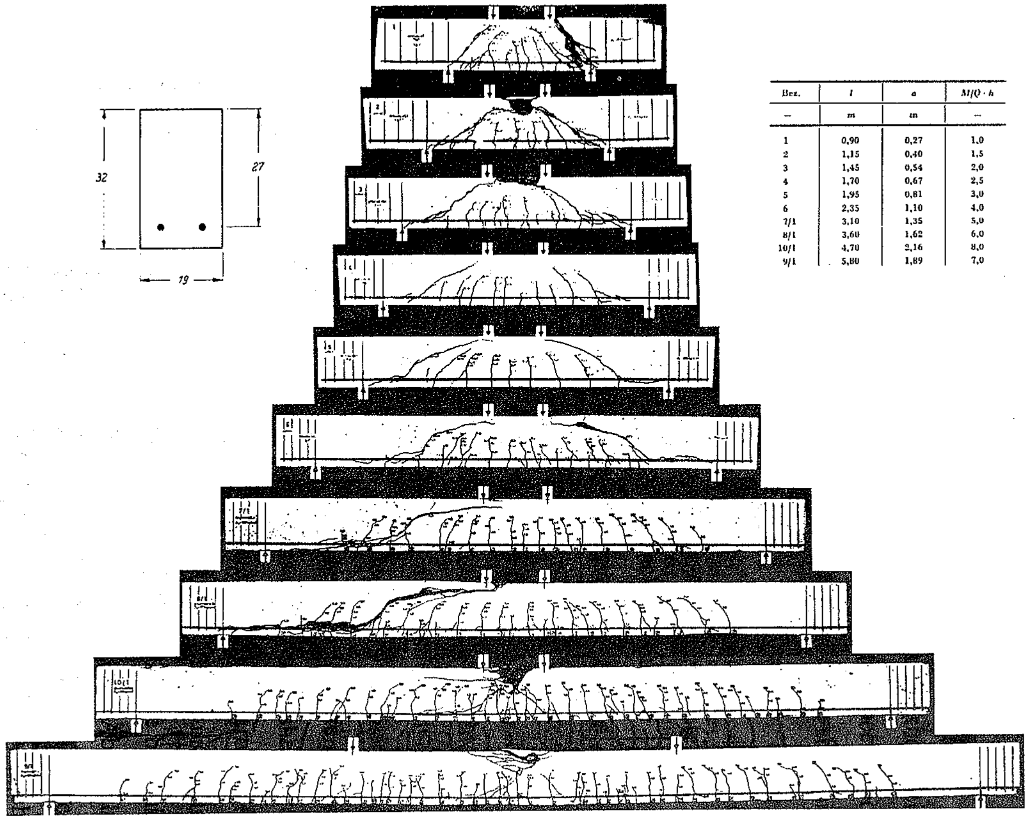
Crack patterns at failure of beam 5.



Crack patterns at failure of beam 6.

d). (continued)

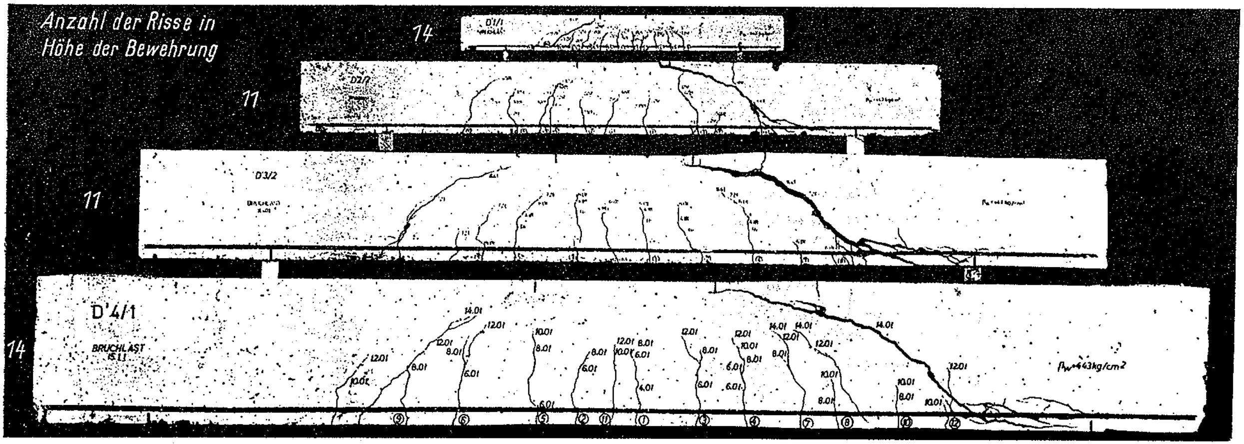
Fig.2.1.2 (continued)



Bez.	l	a	M/Q · h
—	m	m	—
1	0,90	0,27	1,0
2	1,15	0,40	1,5
3	1,45	0,54	2,0
4	1,70	0,67	2,5
5	1,95	0,81	3,0
6	2,35	1,10	4,0
7/1	3,10	1,35	5,0
8/1	3,60	1,62	6,0
10/1	4,70	2,16	8,0
9/1	5,80	1,89	7,0

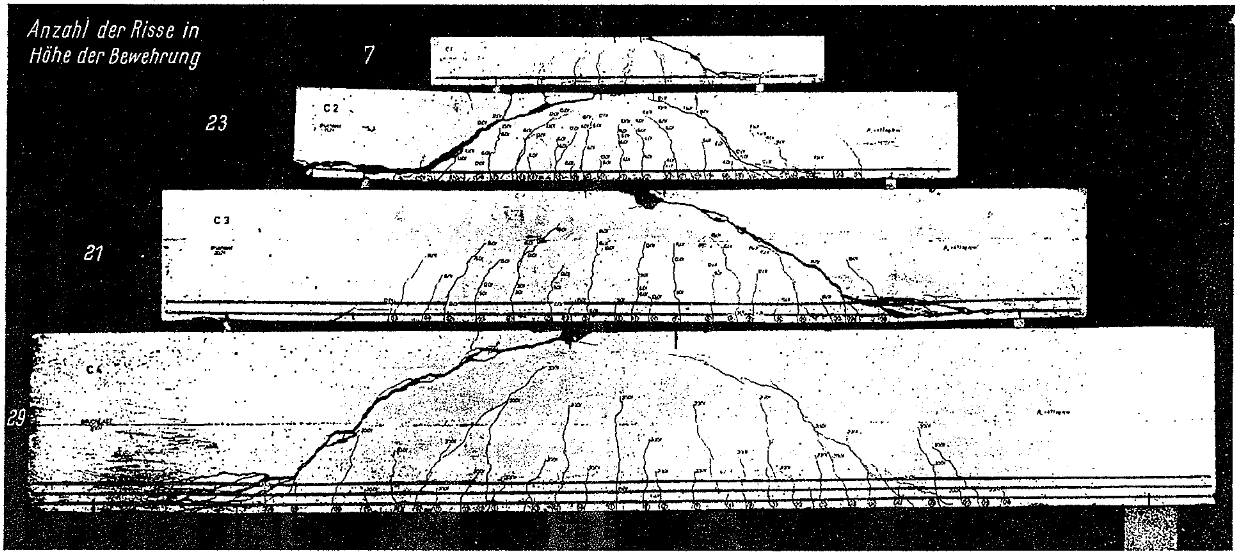
e). Crack patterns of beams at failure tested by Leonhardt & Walther [62,1]

Fig.2.1.2 (continued).



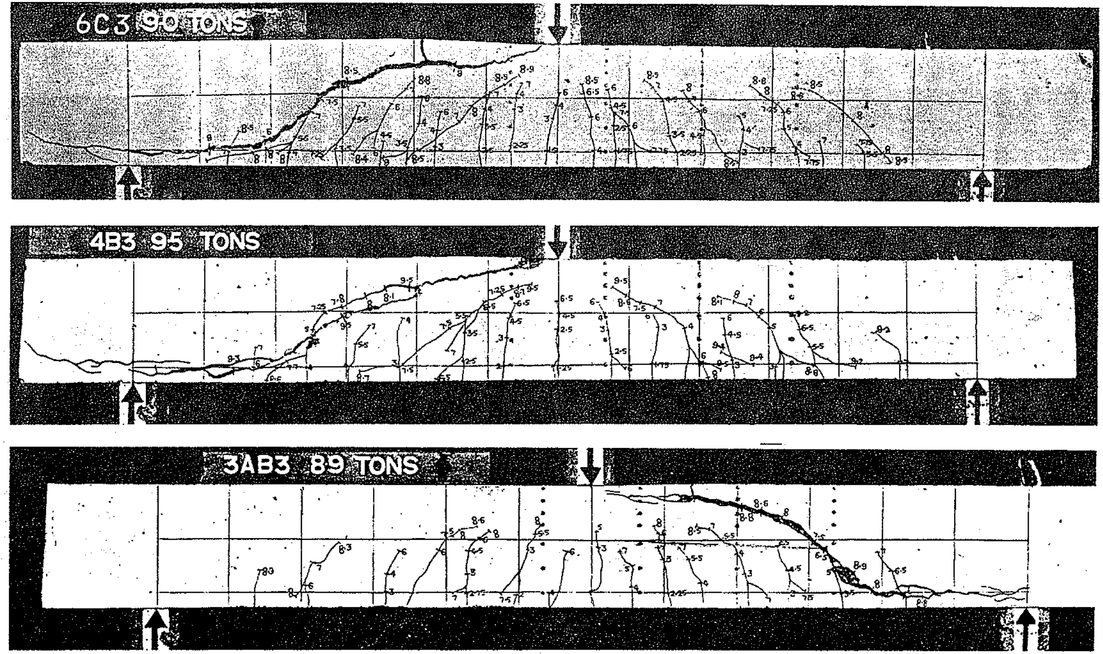
f). Crack patterns of beams with different depths tested by Leonhardt & Walther [62,1]

Fig.2.1.2 (continued)



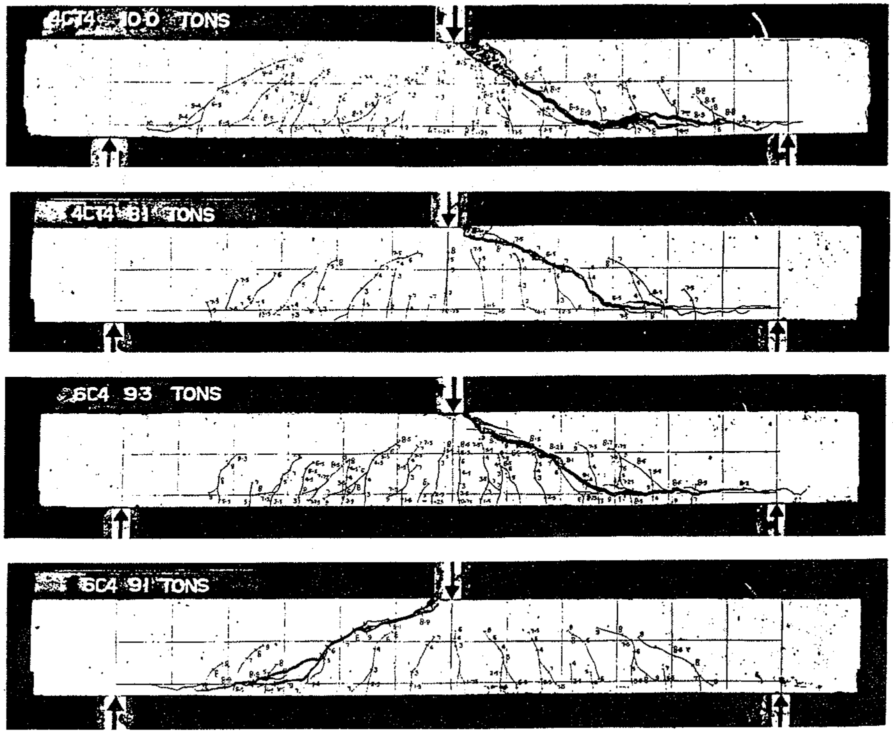
f). (continued)

Fig.2.1.2 (continued)



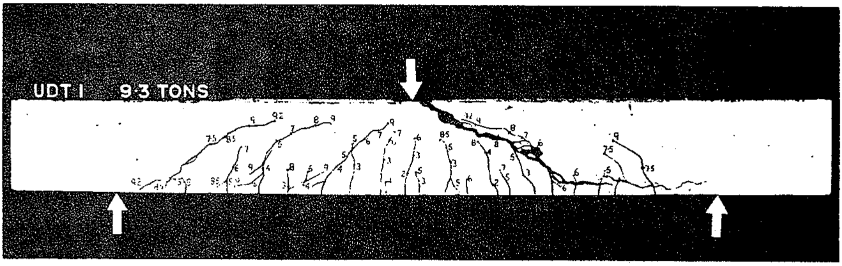
g). Crack patterns of beams at failure tested by Taylor, R.[60,1]

Fig.2.1.2 (continued)

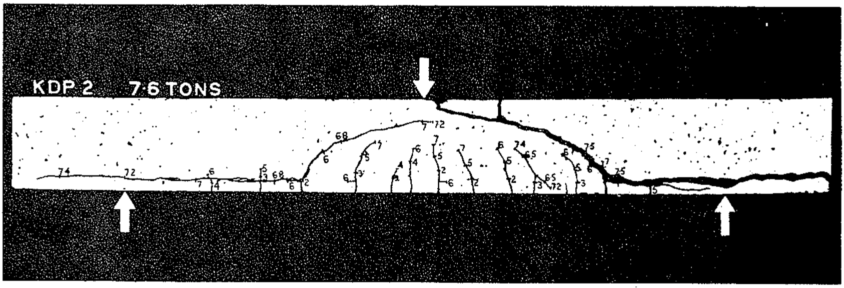


g). (continued)

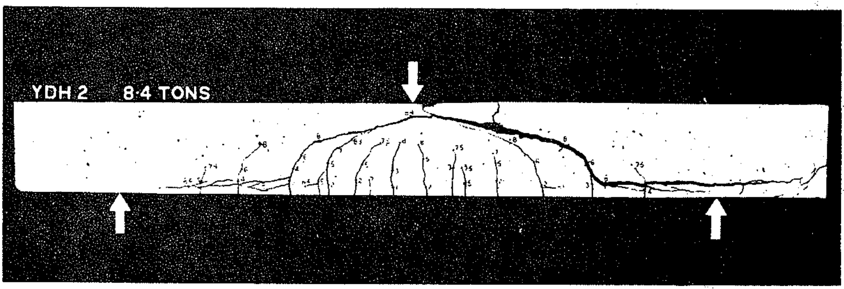
Fig.2.1.2 (continued)



Gravel aggregate, cold-worked deformed bars.



Expanded-clay aggregate, plain mild-steel bars.



Sintered pulverized-fuel ash aggregate, plain mild-steel bars with hooks.

h). Crack patterns of beams with different types of aggregate tested by Taylor,R. [63,2]

Fig.2.1.2 (continued)

and the load at which it occurs is termed the diagonal cracking load. The load at which the specimen collapses is the ultimate load or the load carrying capacity.

Beams might collapse simultaneously with the formation of the critical diagonal crack, or the ultimate load may be higher than the diagonal cracking load. This is due to the fact that cracks are formed with finite distances. This will be explored in more details in the following sections.

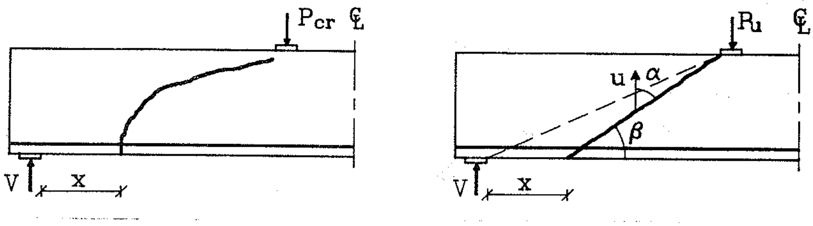
What is the mechanism of this shear failure mode? How to explain the relation between diagonal cracks and shear failure? The answers to these questions should be clear before we continue to find the solution.

2.2 Shear failure mechanism in the theory of plasticity

For many years, researchers have been aware of the function of the critical diagonal crack in shear failure mechanism, and have been trying to find a rational explanation of its formation and its effect on the shear capacity of beams. Due to the complexity of predicting cracks and the relating shear strength, it can be said that up to now, a simple and satisfactory solution has not yet been found.

From the analysis of the shear failure mode, it seems that the theory of plasticity offers a promising opportunity.

Let's consider the beam as shown in Fig.2.1.1. When one of the principal stresses exceeds the tensile strength of concrete, concrete cracks. The cracks are normal to the direction of this principal stress, and the relative displacement



a. Diagonal Crack b. Yield line

Fig.2.2.1 Diagonal crack pattern and yield line in the plastic solution

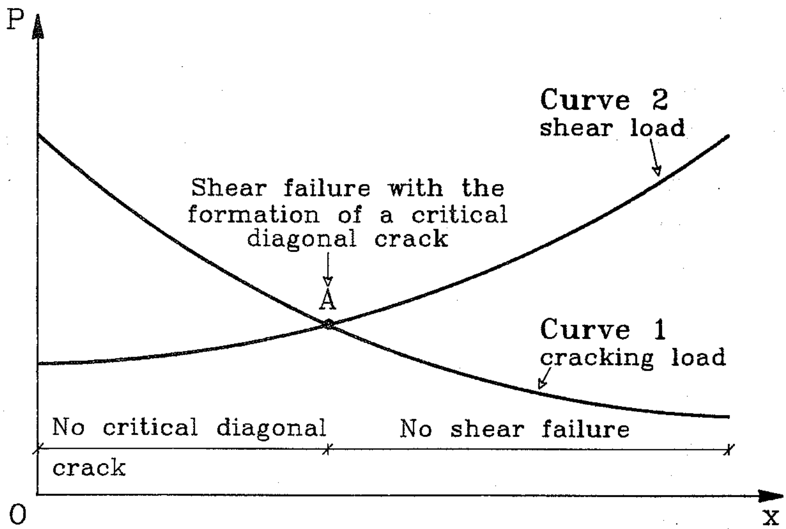


Fig.2.2.2 Cracking load and shear load versus x

along the crack is perpendicular to the crack path. Suppose the starting position of the crack is at a distance x from the support (as shown in Fig.2.2.1(a)). To produce this crack, a certain load is necessary. From bending analysis we know, that the closer the position of the crack is to the support, the higher the load is needed to initiate this crack. The cracking load, corresponding to the formation of a diagonal crack, as a function of the starting position of the crack x is approximately depicted as curve1 in Fig.2.2.2. From tests it is seen that the development of the diagonal cracks stops at a certain distance from the support. Where does the cracking stop?

From observations in experiments, it is quite clear that at failure the relative displacement of the two parts on both sides of the crack is not perpendicular to the path of crack as in a pure separation failure. This fact is confirmed by the measurements of the displacements along the diagonal crack made by Muttoni [90,1], shown in Fig.2.2.3. In plasticity theory terms, this means that sliding failure occurs along the crack.

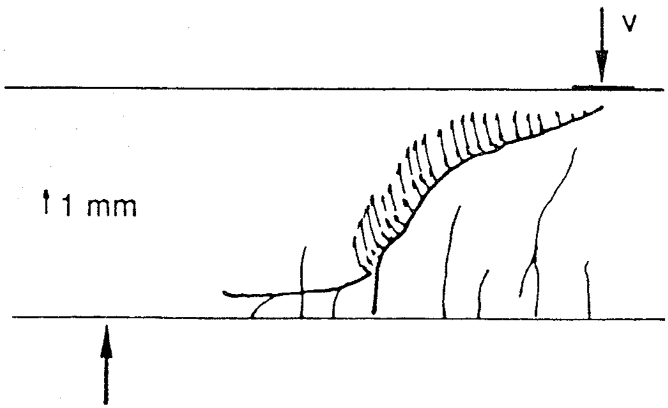


Fig.2.2.3 Displacements along a diagonal crack at different loading stages measured by Muttoni [90,1].

In the plastic theory, concrete is assumed as a rigid plastic material obeying the modified Mohr-Coulomb failure criterion. When the combination of the stresses exceeds this criterion, a sliding failure (in contrast to the separation failure), i.e. the shear failure, takes place along a surface in the space or along a line in the plane. In plane problems, as we are concerned with here, the relative displacement forms an angle with the line. This line is termed a yield line, i.e. a line of discontinuity in displacements. The load found by the plastic theory is the ultimate load or the load carrying capacity.

In the upper bound approach of plastic theory, a simple shear failure mechanism is assumed by setting up a straight yield line as shown in Fig.2.2.1(b). The yield line, which gives the lowest upper bound solution of the load P_0 for the uncracked reinforced concrete beams with zero tensile strength, is the one running from the support to the load shown as a dashed line in Fig.2.2.1(b). For cracked reinforced concrete beams, assuming a yield line following a diagonal crack, the ultimate shear load, corresponding to this yield line, is lower than P_0. In Fig.2.2.1(b) the yield line along a crack is for simplicity depicted as a straight line. The steeper the yield line is, that is the farther the yield line is from the support if the upper end of the yield line is fixed, the higher the shear load becomes. The shear load, corresponding to the yield line in a cracked reinforced concrete beam, as a function of the starting position of the yield line x is shown as curve2 in Fig.2.2.2. Curve1 and curve2 cross at point A where the shear load is equal to the cracking load.

How the shear failure occurs can now be explained. At first, flexural cracks appear in the middle area where the bending moment is maximum. When increasing the load, diagonal cracks form in the shear span. Due to dowel action of the longitudinal reinforcement, interlock action of aggregates and the compression zone of concrete, the shear force is transferred along the cracked section. When the load increases, the development of the diagonal cracks reaches the point where the sliding strength is exceeded along the crack, then the diagonal crack is transformed into a yield line, sliding occurs along the line, and the beam fails in shear.

It is obvious that in uncracked concrete, the yield line will choose the theoretical one which gives the lowest load capacity, but in the case of cracked concrete, failure will take place along the weakest path, i.e. through the crack, instead of going through the uncracked area.

In Fig.2.2.2 it can be seen that when the position of a crack is to the right of point A, then the load causing the crack is lower than the shear capacity of this crack, a yield line can not be formed at this position, and this crack is not dangerous; when the position of a crack is to the left of point A, the shear capacity of the crack is lower than the load causing the crack, shear failure occurs before the critical diagonal crack can be formed at this position. Only at point A the beam fails in shear as a consequence of the formation of a critical diagonal crack. Here the ultimate load in shear is equal to the cracking load, i.e. the diagonal cracking load.

This is the basis of the theory in this paper concerning the shear strength of non shear reinforced concrete beams with critical diagonal cracks.

Apparently, if the load at which the critical diagonal crack occurs and the ultimate load corresponding to a yield line following this crack can be calculated, then both the position of the critical diagonal crack and the shear capacity may be determined.

Since the cracks appear with a finite distance between them, the theoretical diagonal crack will not always be the one really observed. What may happen is illustrated in Fig.2.2.4. Let's imagine that a diagonal crack, originating from point A, has been formed. The theoretical diagonal crack is between this crack and a potential diagonal crack initiating from point B. The load required for producing the existing crack at A can be read off the curve of the cracking load. This load corresponds to point A′. A shear failure can not take place in the beam for this load because the load for producing a shear failure corresponds to point D′ on the curve of the shear load, which is higher than point A′. When the load increases, point C′ on the curve of the shear load is reached. Since no diagonal crack has been formed at this position, the shear capacity, with the yield line going through the uncracked part, is higher than the shear load related to point C′, i.e. no shear failure will take place. When further increasing the load, point B′ on the curve of the cracking load or point D′ on the curve of the shear load might be reached. If point B′ is reached before point D′, a new diagonal crack will be formed at B, and a sudden failure will take place where the load drops down from point B′ to C′. If point D′ is reached before point B′(in the figure D′ is depicted higher than B′), a new diagonal crack can not be developed at B, and a soft failure will take place along the existing crack originating from point A with slowly falling force when deformations are increased.

Thus the theory explains why the shear failure is sometimes very brittle and sometimes more soft.

In both cases the real ultimate load will be higher than the theoretical one.

Depending on whether point B′ or D′ is reached first, the observed position of

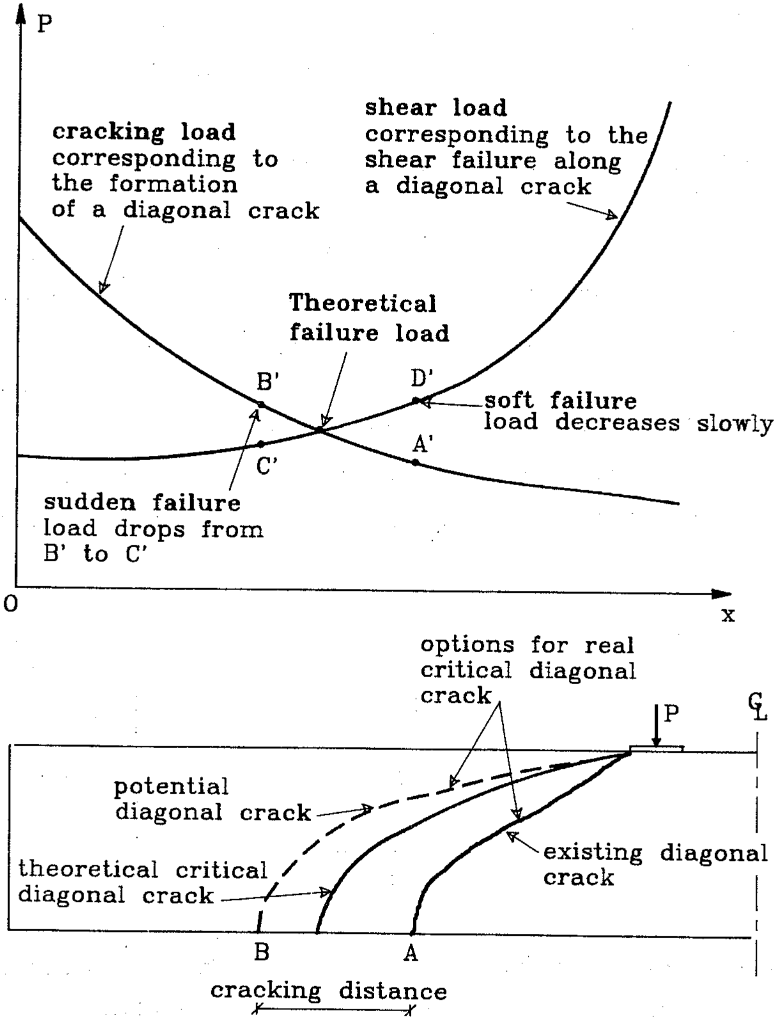
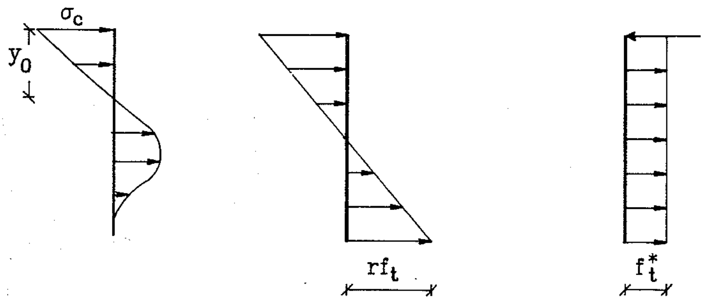

P
cracking load
corresponding to
the formation
of a diagonal crack
shear load
corresponding to the
shear failure along
a diagonal crack
Theoretical
failure load
B'
D'
soft failure
load decreases slowly
C'
A'
sudden failure
load drops from
B' to C'
O
x
options for real
critical diagonal
crack
P
CL
potential
diagonal crack
theoretical critical
diagonal crack
existing diagonal
crack
B
A
cracking distance

Fig.2.2.4

the critical diagonal crack will be closer to or farther from the support, respectively. The real failure line will choose the diagonal crack at A if point D′ is reached first, and the ultimate load will be higher than the diagonal cracking load, otherwise it will choose the potential diagonal crack at B, and the ultimate load will be equal to the diagonal cracking load.

2.3 Maximum cracking moment and diagonal cracking load

In the bending theory of unreinforced concrete beams, the load carrying capacity in bending may be calculated by a moment equilibrium equation for the cracked section if the stress distribution is known. It is assumed that the flexural cracks are perpendicular to the axis of the beam, i.e. vertical. Here the moment corresponding to the load carrying capacity of the section is termed the maximum cracking moment. The real and the equivalent stress distribution corresponding to the state of maximum cracking moment are shown in Fig.2.3.1.



a. Real stress distribution　　b. Elastic equivalent stress distribution　　c. Plastic equivalent stress distribution

Fig.2.3.1 Stress distributions in a section cracked by flexure for the maximum cracking moment

Using the elastic equivalent stress distribution, the maximum cracking moment for a rectangular cross section can be written:

$$M_{cr}=\frac{1}{6}\gamma bh^2f_t \qquad (2.3.1)$$

where γ is a plasticity factor of resistance in the section. It means that γf_t is the flexural modulus and f_t the uniaxial tensile strength of concrete.

The maximum cracking moment alternatively may be found by considering a fully plastic equivalent stress distribution shown in Fig.2.3.1(c). From moment equilibrium, we have

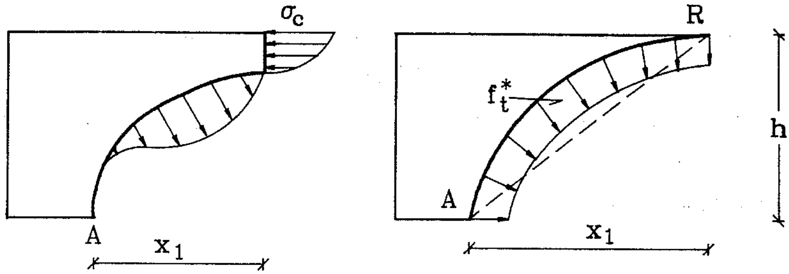
$$M_{cr}=\frac{1}{2}bh^2f_t^* \qquad (2.3.2)$$

Here, f_t^* is the equivalent plastic tensile strength. Putting eq.(2.3.1) equal to eq.(2.3.2) gives

$$f_t^*=\frac{1}{3}\gamma f_t \qquad (2.3.3)$$

The stresses along a curved crack in the shear zone are very difficult to define. Cracking is initiated once the first principal stress σ_1 exceeds the tensile strength of concrete f_t, and the crack is perpendicular to the direction of σ_1. In pure bending, σ_1 is normal to the cross section and the flexural cracks are vertical. In the shear/bending area, the direction of σ_1 is changing due to the combination of shear stress and normal stress, and the crack is curved. A fully developed diagonal crack is illustrated in Fig.2.3.2(a).

This crack propagates gradually, normally initiating at the bottom. The widths of the crack at different points are varying along the crack path, so are the normal stresses.



a. Diagonal crack and stress distribution

b. Plastic equivalent stress distribution

Fig.2.3.2 Diagonal crack and stress distributions for the Maximum cracking moment

In the case of a critical diagonal crack, the crack is supposed to be fully developed. Thus a fully plastic equivalent stress distribution may be assumed, i.e. the tensile stress along the path of crack, perpendicular to the crack, is distributed evenly with a value of f_t^* shown in Fig.2.3.2(b). This assumption is similar to that of a flexural crack (Fig.2.2.1(b)). At the ultimate state, the depth of the compressive zone becomes very small. Neglecting this zone, the upper tip of the diagonal crack is assumed to be at point R at the top surface as shown in Fig.2.2.1.(b).

The direction of the tensile stress is perpendicular to the crack path. Therefore, the internal moment along the curved crack with respect to point R is the same as that along the straight line AR (in Fig.2.3.2(b)). So the maximum cracking moment of a critical diagonal crack becomes

$$M_{cr}=\frac{1}{2}f_t^*bL_{AR}^2 \qquad (2.3.4)$$

Here L_{AR} is the length of the straight line AR, equal to $\sqrt{h^2+x_1^2}$, and f_t^* is the equivalent plastic tensile strength determined by eq.(2.3.3).

In the extreme position of the diagonal crack, i.e. when the diagonal crack turns to be vertical where $L_{AR}=h$, the cracking moment of the diagonal crack calculated by eq.(2.3.4) is the same as that of the flexural crack calculated by eq.(2.3.3) .

For specimens with a rectangular section of standard laboratory size, the plasticity factor of resistance γ is about 1.7. Putting this value into eq.(2.3.3), we find $f_t^* \approx 0.6f_t$. Further reduction of tensile strength due to size effect is considered. Details see section 3.1.1.

From the moment equilibrium of the cracked section, the load that is needed to produce this critical diagonal crack can be calculated, i.e. the diagonal cracking load P_{cr} is determined.

It must be admitted that this way of determining the cracking moment of a diagonal crack is very approximate. However, no better simple procedure seems to be available at present.

Refinements on this point would need a fracture mechanics analysis, following the crack growth and the stress distribution in the crack as the load increases.

2.4 Load carrying capacity of cracked reinforced concrete beam

Take the beam under concentrated loading as an example, see Fig.2.1.1. A shear mechanism formed by a straight yield line following a straight crack is shown in Fig.2.4.1. In the over-reinforced case, the relative displacement u is perpendicular to the reinforcement, i.e. the relative movement of the rigid parts at the sides of the yield line is vertical, $\alpha=90°-\beta$.

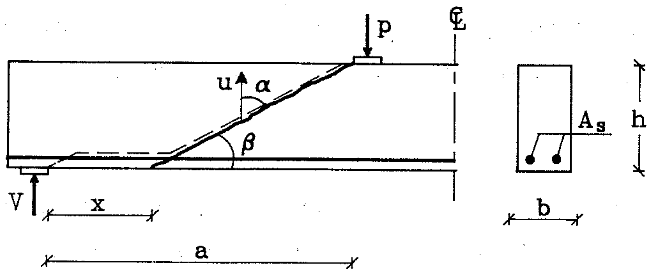


Fig.2.4.1 Shear failure mechanism in the plastic theory

Neglecting the tensile strength a yield line following the dotted line in Fig.2.4.1, which is the one observed in tests, will give the same load carrying capacity.

The internal work of this mechanism is (details see [84,1])

$$W_I = \frac{1}{2} f_c^* b(1 - \cos\beta) \frac{h}{\sin\beta} \cdot u \qquad (2.4.1)$$

Since

$$\cot\beta = \frac{a-x}{h}$$

we have

$$W_I = \frac{1}{2} f_c^* bh\left(\sqrt{1 + \left(\frac{a-x}{h}\right)^2} - \frac{a-x}{h}\right) \cdot u \qquad (2.4.2)$$

The external work is

$$W_E = V \cdot u = P \cdot u \qquad (2.4.3)$$

The work equation $W_I = W_E$ yields the upper bound solution

$$P_u = \frac{1}{2} f_c^* bh\left(\sqrt{1 + \left(\frac{a-x}{h}\right)^2} - \frac{a-x}{h}\right) \qquad (2.4.4)$$

Here $f_c^* = \nu f_c$, where ν is the effectiveness factor for cracked concrete which is different from that for uncracked concrete. This will be discussed in the next section.

The load carrying capacity presented here is an upper bound solution. It is difficult to establish a safe and statically admissible stress field corresponding to a lower bound solution due to the complexity of the stress distribution in a cracked reinforced concrete beam. Therefore the load carrying capacity obtained by lower bound solutions has not yet been searched for.

2.5 Effective compressive strength and effectiveness factor of uncracked and cracked concrete

In the plastic theory of concrete, concrete is assumed as a rigid plastic material. This assumption differs from the real material properties. Also the strength of concrete in the real structure is different from that determined from a small laboratory specimen because the concrete normally is in a micro- or macro-cracked state before failure. Therefore, in the theory of plasticity, the effective compressive strength of concrete f_c^* and the effectiveness factor ν are introduced. We have $f_c^* = \nu f_c$, where f_c is the standard compressive cylinder strength of concrete.

Many tests have been investigated to find the effectiveness factor ν for shear, e.g. [78,1] [88,1]. It turned out that for non shear reinforced concrete beams, ν depends on several parameters.

Based on tests of non shear reinforced concrete beams, in [78,1] ν was found to be

$$\nu = f_1(f_c) \cdot f_2(h) \cdot f_3(\rho) \cdot f_4(\frac{a}{h}) \qquad (2.5.1)$$

Here f_1, f_2... are functions of one variable as indicated. By choosing suitable functions and by means of statistical analysis described in details in [78,1], the four functions were found to be

$$f_1(f_c) = \frac{3.5}{\sqrt{f_c}} \qquad (5 < f_c < 60 \text{ MPa}) \quad (f_c \text{ in MPa}) \qquad (2.5.2)$$

$$f_2(h) = 0.27(1 + \frac{1}{\sqrt{h}}) \qquad (0.08 < h < 0.7m) \text{ (h in m)} \qquad (2.5.3)$$

$$f_3(\rho) = 0.15\rho + 0.58 \qquad (\rho < 4.5\%) \qquad (\rho \text{ in } \%) \qquad (2.5.4)$$

$$f_4(\frac{a}{h}) = 1.0 + 0.17(\frac{a}{h} - 2.6)^2 \qquad (\frac{a}{h} < 5.5) \qquad (2.5.5)$$

The physical meanings of f_1, f_2, and f_3 can be seen clearly.

The dependence on the concrete strength f_c is that increasing f_c leads to decreasing ductility.

The dependence on the depth h is a result of a size effect on the strength. Usually, the strength values are measured on small specimens in laboratory. It is a fact that the larger the size is, the lower the strength will be. Especially for the tensile strength, the size effect is rather strong. It might also be an effect of crack widths, larger beams having larger crack widths and therefore lower sliding strength.

The dependence on the reinforcement ratio ρ describes the effect of ρ on the cracking moment. It might also take account of dowel action of the reinforcement which is neglected in the plastic theory or the fact that increasing ρ leads to decreasing crack widths at failure and eventually increasing the sliding strength.

What is the physical meaning of the dependence on a/h? Various explanations have been given. However the meaning still remains ambiguous and incoherent.

An important fact is neglected in the above analysis, i.e. the effect of the formation of a critical diagonal crack. As mentioned before, within a certain range of shear spans, the shear failure of the beam is always accompanied by the formation of a critical diagonal crack. Hence, the critical diagonal crack plays an important role in the shear failure.

It is an experimental fact that for a non shear reinforced concrete beam, the shear force is transferred by dowel action of the reinforcement, interlock action of the aggregates and by the compression zone of the concrete. If dowel action is neglected or is taken into account by the function $f_3(\rho)$, and if the depth of the compression zone is negligible in the ultimate state, then the shear strength is governed by the interlock action of the aggregates, i.e. the sliding strength. Obviously, the shear force transferred across a cracked section is lower than that transferred across an uncracked section, and the effective strength of cracked concrete is lower than that of the uncracked concrete.

It must be born in mind that even if no visible cracks are formed the strength of the concrete might be reduced because of micro cracking. Micro cracks might be present even before loading due to shrinkage of the cement paste, temperature effects, etc. When we use the term uncracked concrete, it means concrete without visible cracks, i.e. macro cracks. The strength of uncracked concrete might therefore more correctly be termed the strength of micro-cracked concrete. In shear problems the tensile stresses present before cracking will probably reduce the compressive strength because the directions of the micro cracks might be coinciding with the final failure lines due to stress redistribution. Therefore an effectiveness factor must be introduced for uncracked or micro-cracked concrete in shear problems.

Here we denote the effective compressive strength of uncracked or micro-cracked concrete and cracked concrete as f_{c0}^* and f_c^*, respectively.

Normally in the theory of plasticity, the Coulomb criterion is adopted as the sliding failure criterion for concrete. It has the form

$$|\tau| = c - \mu\sigma = c - \sigma \tan\varphi \tag{2.5.6}$$

where τ and σ are the shear stress and the normal stress, respectively, on an arbitrary section. The constant c is called the cohesion, μ the coefficient of friction and φ the friction angle. The normal stress is positive as tension.

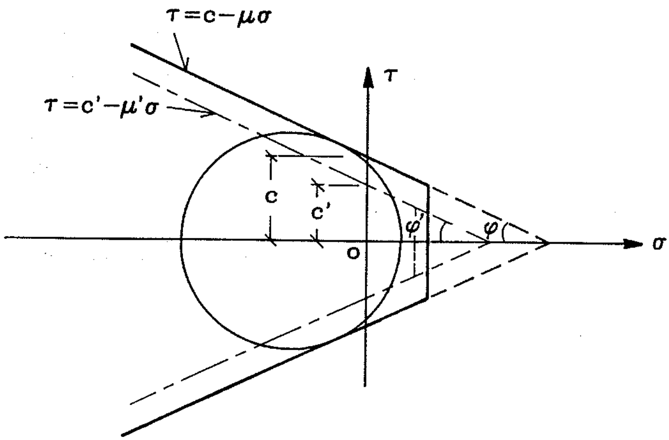


Fig.2.5.1 Modified Coulomb criterion for uncracked (micro-cracked) and cracked concrete

In a σ, τ-coordinate system the failure criterion is shown in Fig.2.5.1. Expressed in principal stresses the sliding criterion becomes

$$k\sigma_1 - \sigma_3 = f_{c0}^* \tag{2.5.7}$$

where

σ_1, σ_3 are the principal stresses, assuming $\sigma_1 > \sigma_2 > \sigma_3$.

$$k = \left(\frac{\cos\varphi}{1-\sin\varphi}\right)^2 = \tan^2\left(\frac{\pi}{4}+\frac{\varphi}{2}\right) = \frac{1+\sin\varphi}{1-\sin\varphi} \tag{2.5.8}$$

$$f_{c0}^* = \frac{2c\cdot\cos\varphi}{1-\sin\varphi} \tag{2.5.9}$$

When the Mohr circle with a combination of certain stresses touches the lines (eq.(2.5.6)), then sliding failure occurs along the yield line. Here the concrete is considered as uncracked or micro-cracked.

If the yield line follows a crack, then obviously, the sliding strength along the cracked interface is lower than that through uncracked concrete.

Tests have shown that when a crack is formed, the tensile strength normal to the crack path becomes very small and negligible, but the cohesion due to the aggregates interlock action still exists. The sliding criterion along the cracked surface can be expressed as

$$|\tau| = c' - \mu'\sigma = c' - \sigma\cdot\tan\varphi' \tag{2.5.10}$$

Here c' and φ' are the cohesion and friction angle for the cracked concrete, respectively.

From tests it seems that the friction angle for normal gravel concrete does not change very much from uncracked to cracked concrete, but the cohesion is lower for the cracked concrete. This means that we may put $\varphi' = \varphi$ and $c' < c$. In the σ, τ-coordinate system, the Coulomb sliding criterion for cracked concrete is shown as the dashed line in Fig.2.5.1.

In order to describe the strength of cracked concrete, we introduce a factor $\nu_s = c'/c$ to take account of the reduction of the cohesion due to cracking. Since the friction angle is not changed, the effective compressive strength is proportional to the cohesion (see eq.(2.5.9)). This results in

$$f_c^* = \nu_s f_{c0}^* \tag{2.5.11}$$

We express the effective compressive strength of uncracked or micro-cracked concrete in the following way

$$f_{c0}^* = \nu_0 f_c \tag{2.5.12}$$

where ν_0 is the effectiveness factor of uncracked or micro-cracked concrete.

In this paper we will primarily be concerned with the strength of macro-cracked concrete and its influence on the shear strength of non shear reinforced beams and slabs. Therefore we will not try to distinguish in a fully rational manner between the strength of micro-cracked concrete and the strength of macro-cracked concrete. We would rather arbitrarily put

$$\nu_0 = \lambda f_1(f_c) \cdot f_2(h) \cdot f_3(\rho) \tag{2.5.13}$$

where f_1, f_2 and f_3 are the same as in eq.(2.5.2) to eq.(2.5.4), and λ is a constant chosen from experiments, and is dependent on the loading type. Of course, other possibilities are open, for instance that the $f_3(\rho)$ function and even the $f_2(h)$ function should be combined with ν_s. This would mean that the sliding strength of a macro crack in shear/bending depends also on the depth of the beam and the reinforcement ratio, which would be quite natural. Further it would mean that the strength of micro-cracked concrete in shear would be taken into account only by the function $f_1(f_c)$, which might also be quite acceptable. For the time being we will proceed by using (2.5.13).

From eq.(2.5.11) and eq.(2.5.12), the effectiveness factor and effective compressive strength for the cracked concrete can be expressed as

$$\nu = \nu_s \cdot \nu_0 \quad (2.5.14)$$

$$f_c^* = \nu f_c \quad (2.5.15)$$

The factor ν_s, and maybe even the functions f_2 and f_3 are parameters which reflect the reduction of sliding strength along the cracks. The reduction of strength might be determined from tests of cracked specimens, such as the push off or pull off tests although they do not include the effect of bending (this will be investigated in the following reports).

The sliding strength along a crack depends on many parameters, e.g. the width of the crack, the size and strength of the aggregates, the roughness of the crack faces, and the confining conditions of the crack, etc. Tests show that when the initial widths of the cracks are within a certain range, say less than 0.5mm, the value of ν_s varies in a limited range. Analysis in [78,1] shows that for the specimens of the push off type, the value of ν_0 is about 0.7, and $\nu_0\nu_s$ about 0.45 which means that ν_s is about 0.67.

At present stage, the value of ν_s for shear/bending is chosen as a constant 0.5, disregarding the fact that ν_s is far from being a constant. More research is needed before the value can be more accurately stated.

2.6 General assumptions

In this report, we will only be concerned with the shear strength of non shear reinforced concrete beams disregarding the cracks caused by non external load such as creep, shrinkage, etc. The general assumptions are:

1. Concrete is assumed to be a modified Coulomb material with a zero tension cut-off, which means that the tensile strength is ignored in the plastic solutions in this paper.
2. Plane stress field is assumed in the analysis.
3. Only rectangular sections are considered.
4. Dowel action of the reinforcement is ignored.
5. The beams are over reinforced, i.e. the tensile reinforcement does not yield at failure.

III. SHEAR CAPACITY OF CONVENTIONAL REINFORCED CONCRETE BEAMS, DEEP BEAMS, CORBELS AND PRESTRESSED BEAMS WITHOUT SHEAR REINFORCEMENT

In this section, specimens without shear reinforcement from different test series are investigated.

3.1 Shear capacity of conventional beams subjected to concentrated loading

First, we consider a simply supported beam loaded by two symmetrical concentrated forces.

The beam is over reinforced with a reinforcement area A_s along the bottom. Details can be seen clearly in Fig.3.1.1. It is assumed that no anchorage failure occurs.

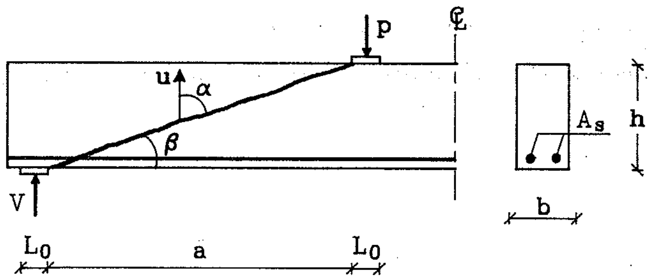


Fig.3.1.1 Beams subjected to concentrated loading. The yield line for the lowest upper bound solution in the usual plastic theory

Here a is the clear shear span defined as the distance between the edges of the support platen and the load platen. L_0 is the length of the load and the support platen (for simplification the length of both platens is assumed to have the same value).

3.1.1 Shear capacity according to the usual plastic theory

In an upper bound approach of the usual plastic theory which does not take the formation of a critical diagonal crack into account, a straight yield line is assumed and the lowest upper bound solution of the shear capacity of an over-reinforced beam without shear reinforcement is found to be

$$\frac{\tau_0}{\nu_0 f_c}=\frac{1}{2}\left(\sqrt{1+\left(\frac{a}{h}\right)^2}-\frac{a}{h}\right) \qquad (3.1.1.1)$$

Here the shear strength τ_0 is normalized by being divided by the effective compressive strength $\nu_0 f_c$ to facilitate the comparison with tests. This solution is referred to as the original plastic solution in the following sections. In this solution, the yield line runs from the edge of support platen to the edge of load platen. Details see [84,1].

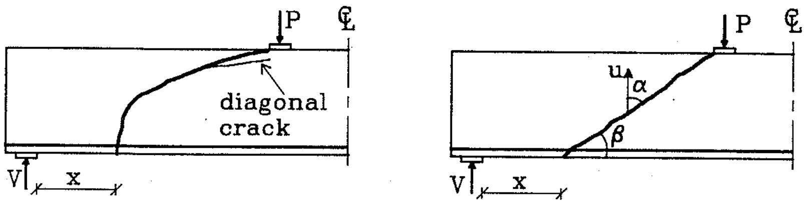
From analysis of the available test results, considering the effect of the critical diagonal crack, the effectiveness factor of the micro-cracked concrete ν_0 is chosen to be

$$\nu_0=1.6f_1(f_c)\cdot f_2(h)\cdot f_3(\rho) \qquad (3.1.1.2)$$

The functions f_1,f_2,and f_3 are determined by eq.(2.5.2) to eq.(2.5.4).

3.1.2 Shear capacity according to plastic theory with a critical diagonal crack

In a beam with a critical diagonal crack, the crack path and the final failure line, i.e. the yield line, are curved as illustrated in Fig.3.1.2(a).It is observed that in the lower part of the beam the yield line coincides with the critical diagonal crack. Close to the load platen, sometimes the yield line follow the crack to the edge of the load platen, and sometimes the two curves depart and the yield line ends at the outside edge of load platen, while the critical diagonal crack tends to continue to a point beyond the platen. In general, it can be assumed that the yield line coincides with the critical diagonal crack to the edge of loading platen.



a. Critical diagonal crack and yield line

b. Simplified critical diagonal crack and yield line

Fig.3.1.2.1 Critical diagonal crack and yield line

Theoretically there is no difficulty in determining the internal work dissipated along an arbitrary curved yield line. However, to simplify the calculations, the curved yield line will be replaced by a straight yield line which is assumed to follow a straight critical diagonal crack path initiating from the bottom of the beam as shown in Fig.3.1.2(b).

The shear capacity P_u then can be calculated by eq.(2.4.4). The nominal shear strength is denoted by $\tau_u = V_u/bh = P_u/bh$. Here, the effectiveness factor of cracked concrete is $\nu = \nu_s \cdot \nu_0$, the effective compressive strength of concrete is

$f_c^* = \nu f_c$. In order to compare with the original plastic solution, the shear strength is normalized with regard to the effective strength of micro-cracked concrete. This yields

$$\frac{\tau_u}{\nu_0 f_c} = \frac{1}{2}\nu_s\left(\sqrt{1+\left(\frac{a-x}{h}\right)^2} - \frac{a-x}{h}\right) \qquad (3.1.2.1)$$

In this formula, x represents the starting position of the critical diagonal crack, which will be determined in the following section.

3.1.3 Shear capacity determined by the diagonal cracking load

Typical crack patterns are shown in Fig.2.1.2.

Kani suggested that all the inclined cracks in the shear/bending zone converge to a point O as shown in Fig.3.1.3.1 (details see [64,1]), Here y_0 is the depth of the compressive zone when the first diagonal crack occurs. The position of point O varies with many parameters and is not easy to define.

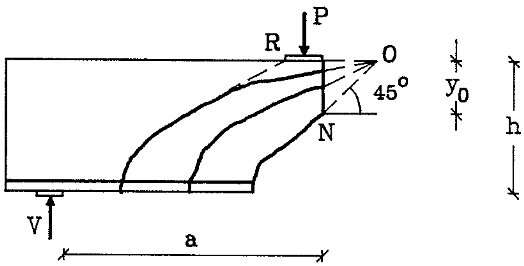


Fig.3.1.3.1 Patterns of the diagonal cracks suggested by Kani [64,1]

When the critical diagonal crack is fully developed, and the specimen is about to fail or close to fail in shear, then the depth of the compressive zone will become small. Thus it will make little difference to change the convergent point for the last diagonal crack to point R at the edge of the load platen thereby neglecting the compressive zone (see Fig.3.1.3.1(b)).

In Fig.3.1.3.2, the part at the side of the support is shown isolated. As assumed in section 2.3, the maximum cracking moment M_{cr} along the curved crack is the same as that along the straight line, and can be calculated by eq.(2.3.2).

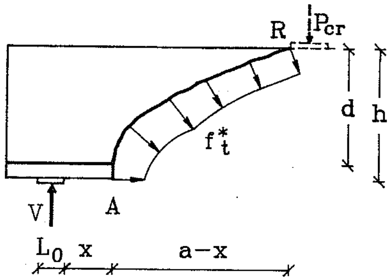


Fig.3.1.3.2 State of forces and stresses for the maximum cracking moment.

From moment equilibrium about point R, we have

$$V\cdot(\frac{1}{2}L_0+a)=M_{cr} \tag{3.1.3.1}$$

where

$$V=P_{cr}$$

$$M_{cr}=\frac{1}{2}f_t^{*}bL_{AR}^{2}$$

$$L_{AR}=\sqrt{h^2+(a-x)^2}$$

Putting these values into eq.(3.1.3.1), the diagonal cracking load becomes

$$P_{cr}=\frac{1}{2}f_t^*bh\frac{1+(\frac{a-x}{h})^2}{\frac{a}{h}+\frac{1}{2}\cdot\frac{L_0}{h}} \tag{3.1.3.2}$$

The nominal shear stress at cracking is defined as $\tau_{cr}=V_{cr}/bh=P_{cr}/bh$, so the normalized shear strength determined by the diagonal cracking load is

$$\frac{\tau_{cr}}{\nu_0 f_c}=\frac{1}{2}\frac{f_t^*}{\nu_0 f_c}\cdot\frac{1+(\frac{a-x}{h})^2}{\frac{a}{h}+\frac{1}{2}\cdot\frac{L_0}{h}} \tag{3.1.3.3}$$

As discussed in section 2.3, the value of f_t^* for small laboratory specimens is taken as $0.6f_t$, where f_t is the tensile strength of concrete. According to tests performed for normal strength concrete, the value of f_t may be taken to be

$$f_t=0.26f_c^{\frac{2}{3}} \qquad (f_t,\ f_c \text{ in MPa}) \tag{3.1.3.4}$$

To include large beams in the analysis, size effects must be taken into account. From reference [54,1], the size effect function is chosen as

$$s(h)=(\frac{h}{0.1})^{-0.3} \qquad \text{(h is the depth of beam in m)} \quad (3.1.3.5)$$

This function shows that for a 0.1m deep beam, the tensile strength of the concrete is f_t, whereas for a 1m deep beam, the tensile strength is about $0.5f_t$.

This formula is based on splitting tests made by Forsell[54,1]. The Weibull root describing the size effect is found to be 0.27, which is here rounded up to 0.3.

The tensile strength for a beam with the depth h becomes

$$f_t=0.26f_c^{\frac{2}{3}}\cdot s(h) \qquad (3.1.3.6)$$

3.1.4 Starting position of the critical diagonal crack and shear capacity

As stated above, the ultimate load equals the diagonal cracking load when the critical diagonal crack is formed. The equation $P_u=P_{cr}$ leads to

$$f_c^*(\sqrt{1+(\frac{a-x}{h})^2}-\frac{a-x}{h})=f_t^*\frac{1+(\frac{a-x}{h})^2}{\frac{a}{h}+\frac{1}{2}\cdot\frac{L_0}{h}} \qquad (3.1.4.1)$$

A simple analytical solution is not easy to obtain. Instead, a numerical method is used to find the value of x/h. In this way, the starting position of the critical diagonal crack for given a/h is found, and consequently the ultimate load, i.e. the shear capacity, can be determined.

3.1.5 Parameter studies

In the above deductions, the influence of the variables on the theoretical results is not easy to evaluate. Therefore, the values of main parameters, such as f_c, h, and ρ are varied in the calculation to find their effects.

The default values of the variables in the calculation are set to

$$f_c=25MPa \quad h=30cm \quad \rho=2\% \quad \nu_s=0.5$$

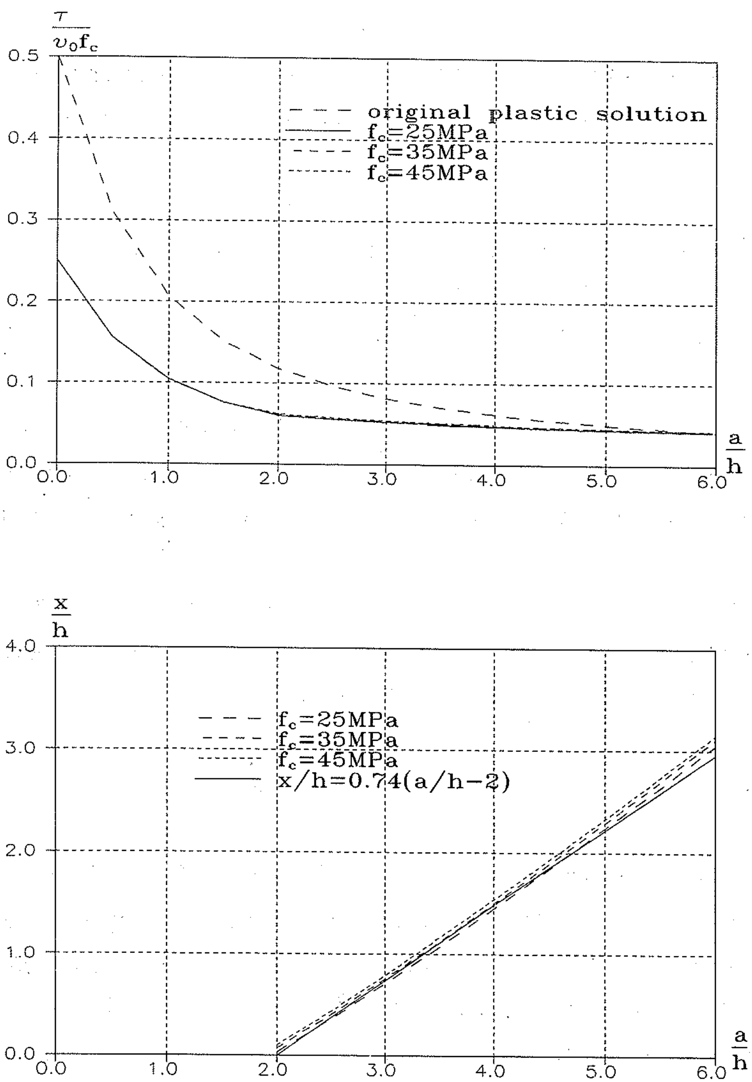
The length of the load and the support platen L_0 is taken as the minimum value derived from the stress field of the lower bound solution, i.e. the value giving the compressive strength νf_c beneath the loading platen. Details see [81,1].

Only one variable is changed at one time. The results are shown in Fig.3.1.5.1. In the diagrams the original plastic solution without taking into account the critical diagonal crack is also plotted for comparison.

If x is negative in the calculation, then zero is assigned to x.

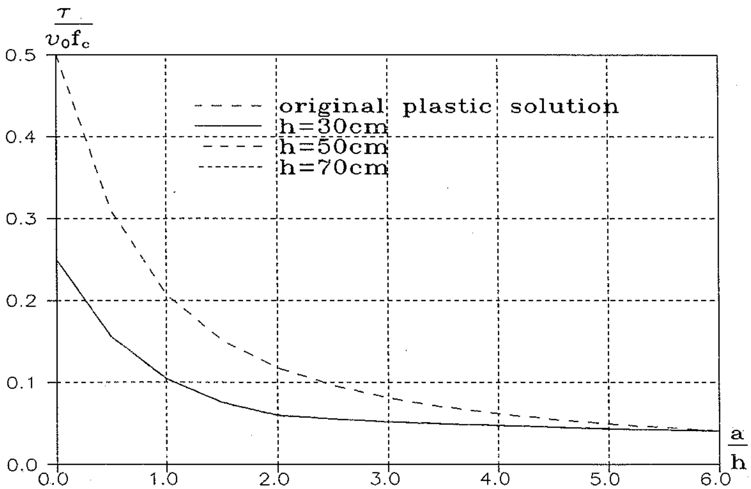
The changes of the parameters are found to have little effect on the results of the normalized shear strength. The position of the critical diagonal crack is changed within a very limited range. This makes the comparison between theoretical calculations and test results rather convenient. Therefore, in the following theoretical calculations, the default values of f_c, h, and ρ are used.

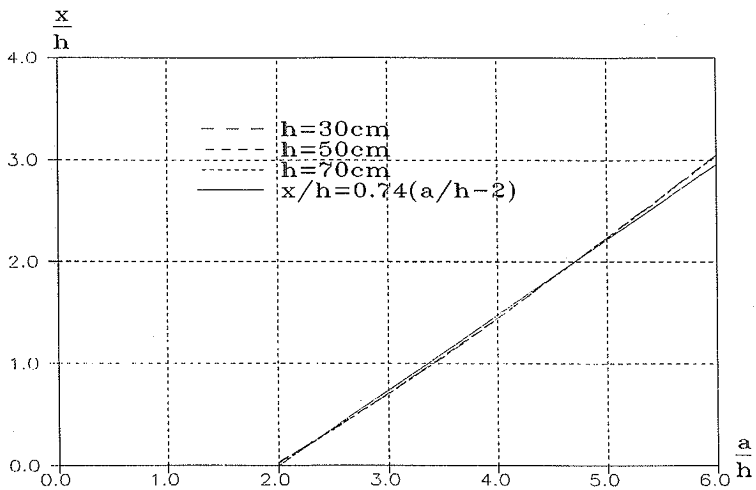
The present calculation is not valid beyond the point where the curves of the original plastic solution and the new calculation cross.



a). Shear strength and x/h for different values of f_c

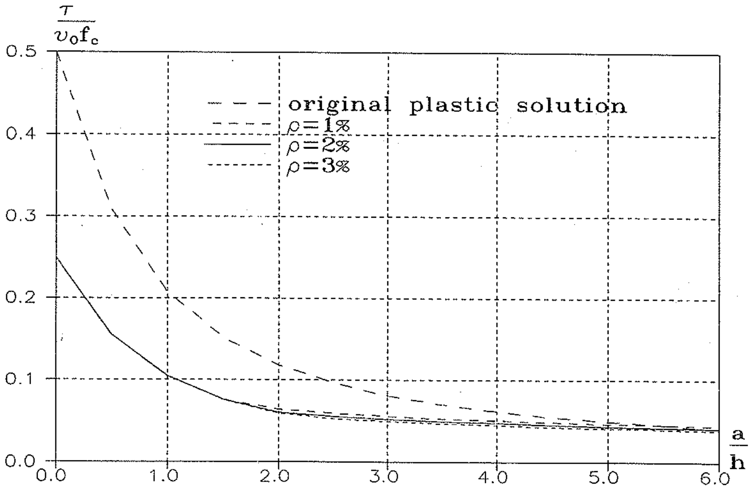
Fig.3.1.5.1 Shear strength and x/h for different values of the parameters

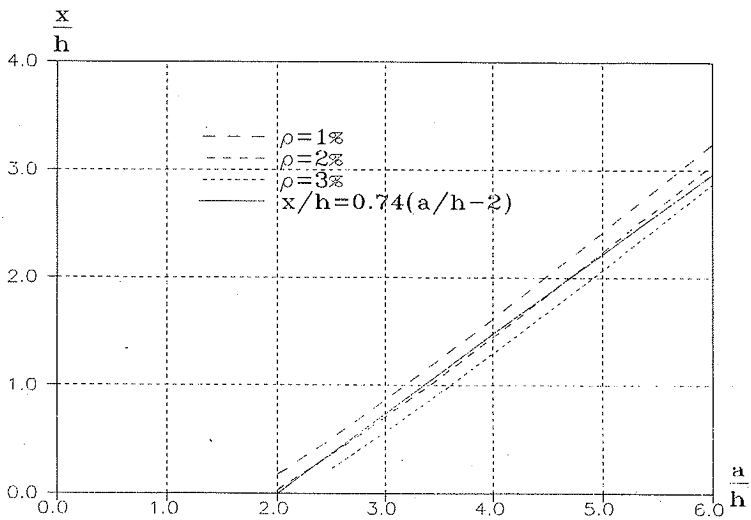




b). Shear strength and x/h for different values of h

Fig.3.1.5.1 (continued).





c). Shear strength and x/h for different values of ρ

Fig.3.1.5.1 (continued).

From the calculation, the value of x is found approximately to be

$$\frac{x}{h} \approx 0.74(\frac{a}{h} - 2.0) \qquad (\frac{a}{h} \geq 2.0) \qquad (3.1.5.1)$$

$$\frac{x}{h} = 0 \qquad (\frac{a}{h} < 2.0) \qquad (3.1.5.2)$$

3.1.6 Experimental verification

As mentioned above, only the over reinforced beams from different test series failing in shear are treated. The criterion for over reinforcement is

$$\Phi \geq \frac{1}{2} \nu_s \cdot \nu_0 \qquad (3.1.6.1)$$

$$\Phi = \frac{A_s f_Y}{bhf_c} \qquad (3.1.6.2)$$

Here Φ represents the degree of reinforcement, and f_Y is the yield strength of the reinforcement.

First, 74 beams from the tests by Leonhardt[62,1] and Krefeld [66,1] are treated. For comparison, the original plastic solution is plotted as well. The results are shown in Fig.3.1.6.1.

It can be seen that the values of the shear strength of conventional reinforced concrete beams with $a/h \geq 2$ from both test series are in good agreement with the present calculation. Fig.2.1.2 shows clearly that within this range of shear span ratios, the beams fail in shear with the formation of a critical diagonal

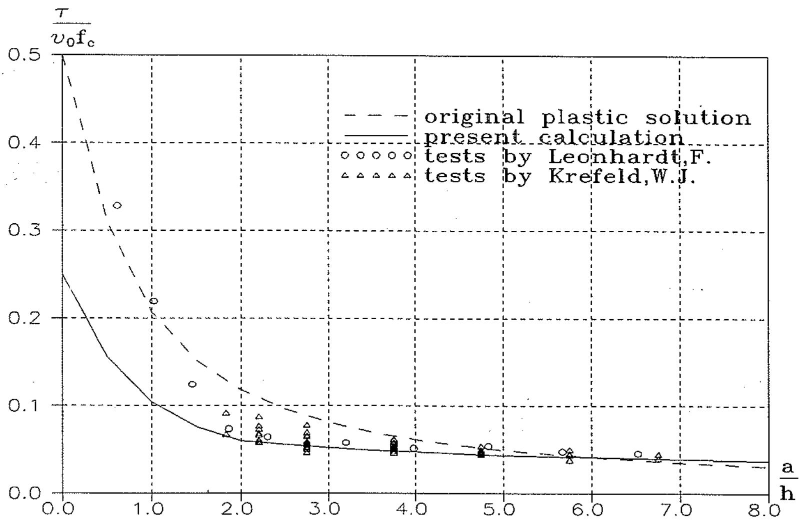


Fig.3.1.6.1 Comparison of theoretical shear strength and test results [62,1][66,1]

crack.

In Fig.3.1.6.2 and Fig.3.1.6.3 the theoretical starting positions of the critical diagonal cracks are compared with those found for the specimens in [62,1] and [66,1]. It is known that crack distance depends on the type of reinforcement, bond conditions, environmental actions, loading, etc. As stated in section 2.2, the theoretical diagonal crack is not always the one observed in experiments. Depending on the relative position of the theoretical critical diagonal crack and the possible nearest diagonal cracks, the real position of the critical diagonal crack is $\pm L_{cr}$ or $\pm L_{max.}$ around the theoretical point, where L_{cr} represents the mean distance between flexural cracks, and $L_{max.}$ the maximum distance.

Approximately $L_{max.}=2L_{cr}$. It is recommended in [83,1] to calculate L_{cr} by the formula

$$L_{cr}=47+0.61\alpha \qquad (L_{cr} \text{ in mm}) \qquad (3.1.6.1)$$

$$\alpha=\frac{1.7W_{uy}}{\pi\sum_{j=1}^{n}d_jh_i}$$

where

$$W_{uy}=\frac{1}{6}bh^2 \text{ for a rectangular section}$$

d_j is the diameter of longitudinal steel bars, n is the number of the bars, and h_i is the lever arm, i.e. the distance between the tensile and compression resultant. Details see [83,1].

Take the tests by Leonhardt, where the average $f_c=29$MPa, $b\times h=19\times 32$cm, $h_e=27$cm, $h_i\approx 0.9h_e=24.3$cm, and the longitudinal reinforcement consisting of 2 bars with the diameter of 26mm. The mean cracking distance determined by eq.(3.1.6.1) is $L_{cr}/h=0.41$ and the maximum cracking distance is $L_{max.}/h=0.82$. Fig.3.1.6.2 shows the comparison with the test results from [62,1].

The comparison of x/h from the test results by Krefeld [66,1] is depicted in Fig.3.1.6.3. For example, for a beam with $b\times h=6\times 12$in, $h_e=9.4$in, $h_i\approx 0.9h_e=8.46$in, and the reinforcement consisting of 2-#9, we have $L_{cr}/h\approx 0.42$, $L_{max.}/h=0.84$.

It can be concluded that the scatter of the starting position of critical cracks may be explained by the finite crack distances.

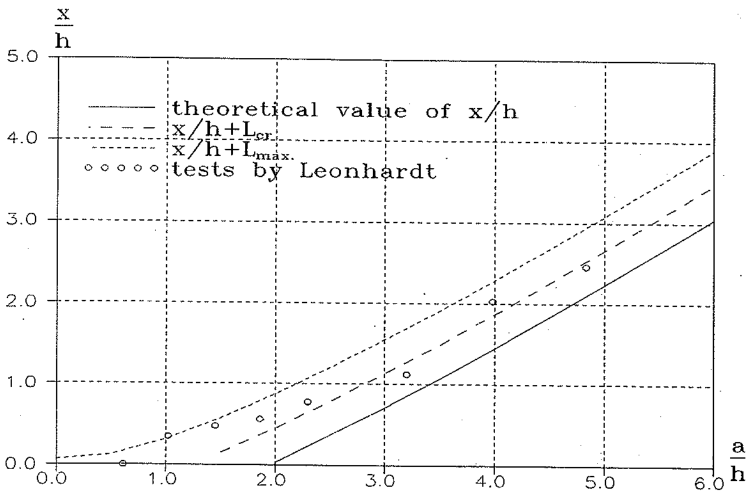


Fig.3.1.6.2 Comparison of x/h with test results by Leonhardt [62,1].

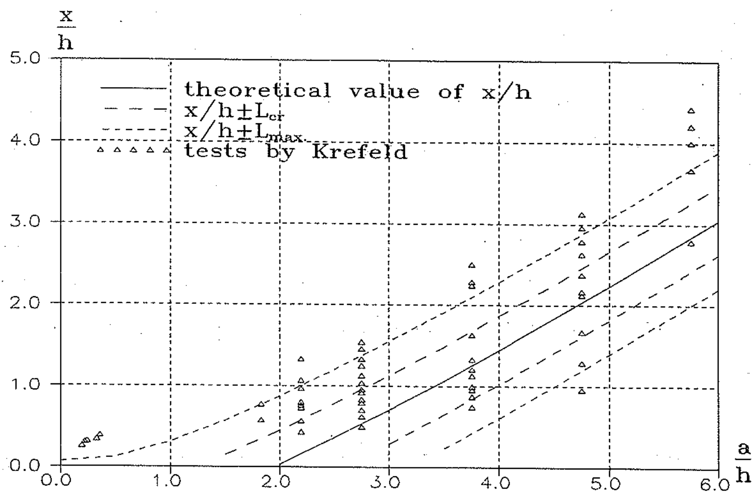


Fig.3.1.6.3 Comparison of x/h with test results by Krefeld [66,1]

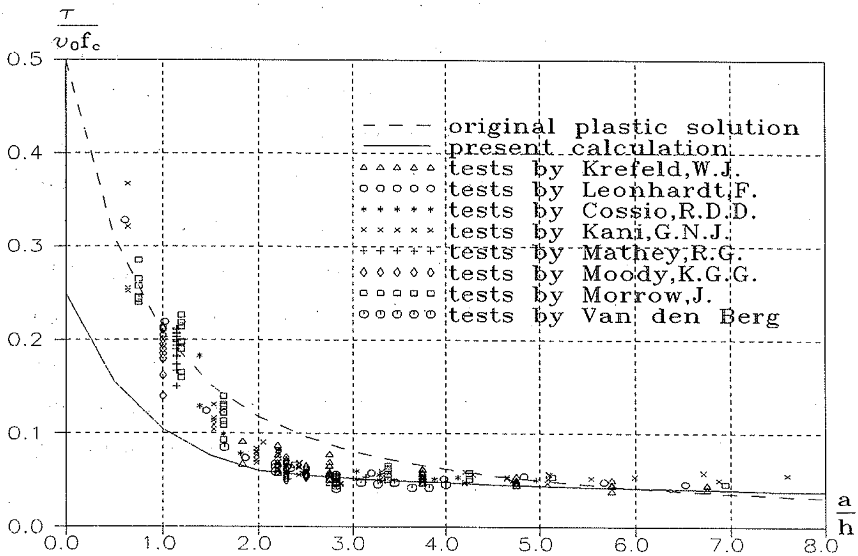


Fig.3.1.6.4 Comparison of theoretical shear strength and test results from [66,1][62,1][60,2][64,1][63,1][54,2][57,1][62,2].

In Fig.3.1.6.4 the results of 280 beams from 8 test series are compared with the calculated shear strength. It shows:

(1). For beams with a/h approximately larger than 2.0, the test results are in rather good agreement with the present calculation. As observed in the experiments, in this range of the shear spans, the shear failure is always featured by the formation of a critical diagonal crack.

(2). When $a/h \leq 0.75$, then $x=0$ (as stated in eq.(3.1.5.2)), and the angle β becomes $\beta \geq 53°$. Consequently the angle α, which is the angle between the relative displacement and the yield line, becomes $\alpha=90°-\beta\leq 37°$ (see Fig.3.1.2.1)). For normal gravel concrete the friction angle φ is about 37°. Therefore in this case $\alpha\leq\varphi$. When the shear span is getting smaller, the width of the beam compared with the shear span becomes larger, and the effect of plane strain begins to dominate the load carrying capacity, meaning that $\alpha\leq\varphi$ is not possible. We know from the plastic theory that for $\alpha\geq\varphi$ the formulae of the load carrying capacity in a plane strain field are identical to those in a plane stress field. But when α is found to be less than φ, the restraints imposed by the plane strain field will raise the ultimate load compared to that determined under the assumption of a plane stress field. This is the reason why the tests with a/h ≤ 0.75 are lying above the original plastic solution deduced on the basis of a plane stress field. In this region no visible critical diagonal crack initiating from bending is observed in the experiments.

(3). For beams with a/h between 0.75 and 2, the test results almost fill out the space between the curve corresponding to the load carrying capacity for the yield line following a crack and the curve corresponding to the original plastic solution. For a/h<2, the diagonal cracking load is lower than the shear strength determined by plastic theory for the yield line following a diagonal crack as shown in Fig.3.1.6.5. This means that the starting position of the crack is found to be $x<0$. In this case we put $x=0$ in the present calculation corresponding to a critical diagonal crack starting from the support (line RA in Fig.3.1.6.6), and the shear strength τ_u determined by the present calculation corresponds to point A_0. However, as mentioned before, the cracks in reality are formed with finite distances, termed as cracking distance or bond length, between them. If the distance between an existing crack near the support and the support is not long enough to make a new crack, then the final yield line will either follow the existing crack or go through uncracked concrete following a line from support

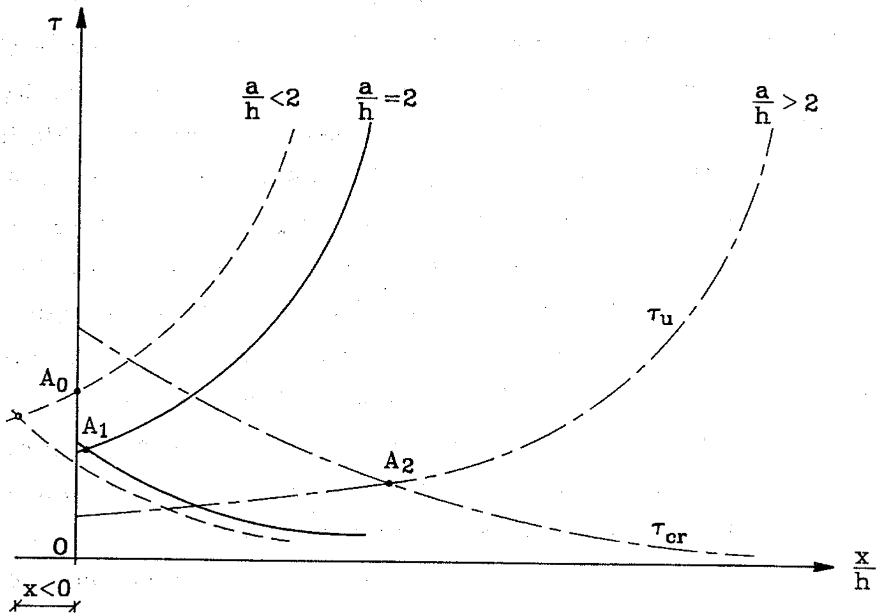


Fig.3.1.6.5 Diagonal cracking load and ultimate load as function of x/h for different a/h

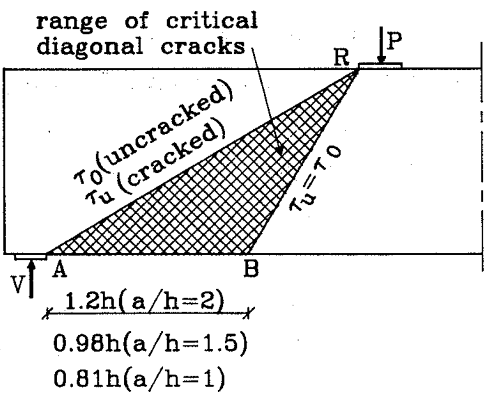


Fig.3.1.6.6 Possible positions of critical diagonal cracks for a/h<2

to loading point. In Fig.3.1.6.6, the line RB is a yield line following a crack at a position which gives the same value of shear strength τ_0 as in the original plastic solution corresponding to a yield line from support to load. The possible position of a yield line coinciding with a crack is between line RA and RB. If the nearest existing crack is between RA and RB, and the length from this crack to the support is not long enough to make a new crack, then the yield line will follow the existing crack. If the length is long enough, a new crack will be formed, and the yield line will follow this new crack. If the nearest existing crack is beyond the line RB and the distance from this crack to the support is less than the bond length needed to make a new crack, then the new crack can not be formed, and the yield line will go through uncracked concrete from support to load, otherwise a new crack will be developed. In any case the load carrying capacity is greater than the present calculation. We therefore may state that if a crack can not be developed between line RA and RB, then the load carrying capacity is close to the original plastic solution. If a crack may be formed between line RA and RB, the load carrying capacity will be between the original plastic solution and present calculation. The distance AB for $a/h<2$ is found to be about of the same order of magnitude as the distance between cracks, the mean cracking distance being of the order $0.3 \sim 0.5h$ and the maximum cracking distance being approximately twice as big. Therefore any point between A and B will be a probable origin of a crack and consequently the load carrying capacity may be found anywhere between the original plastic solution and the present calculation.

Similar tendency of the load carrying capacity can be found as well for beams with $a/h \geq 2$, due to the scatter of the position of the critical diagonal cracks.

As it appears from Fig.3.1.6.5 that the scatter of the formation of the cracks has much more influence on the load carrying capacity than for a/h>2. This is probably the main reason for the large span of shear strengths found in this range of shear spans.

Other cracks may be formed as well in this range, the most likely one being a splitting crack running from load to support. The formation and the effect of a splitting crack or other cracks in this range of shear spans will not be treated in details in this paper. We only state that the yield line will not follow a splitting crack completely, a part of it in the ends will go through a zone of uncracked concrete. This will increase the value of the ultimate load. When the length of the yield line is large, the relative increase of the ultimate load is small and negligible. But when the shear span is getting smaller and the length of the yield line becomes shorter, the effect of this increase will be obvious and the ultimate load will be raised considerably. This is why even if a crack is formed, the shear strength is still higher than the present calculation, i.e. higher than ν_s times the original plastic solution.

Comparison of the ultimate load determined by the present calculation and the test results with $a/h \geq 2$ is shown in Fig.3.1.6.7. The mean value of the ratio $\tau_{test}/\tau_{cal.}$ is 1.10, the standard deviation is 0.14, and the coefficient of variation is 0.13.

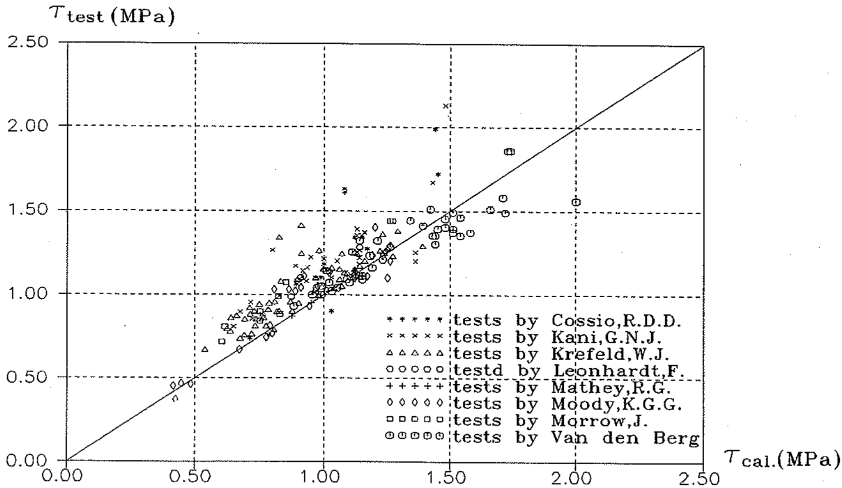


Fig.3.1.6.7 Comparison of shear strength obtained by theoretical calculation and by tests for $a/h \geq 2$.

As stated in section 3.1.5, the starting position of the critical diagonal crack may be estimated approximately by

$$\frac{x}{h}=0.74\left(\frac{a}{h}-2\right) \qquad \left(\frac{a}{h}\geq 2\right) \tag{3.1.6.2}$$

Thus the shear capacity may be determined by eq.(3.1.2.1) easily. In Fig.3.1.6.8 the shear strength determined by eq.(3.1.2.1) with x estimated by eq.(3.1.6.2) is compared with the same set of tests as above. The mean value of the ratio $\tau_{test}/\tau_{cal.}$ is 1.08, the standard deviation is 0.14, and the coefficient of variation is 0.13.

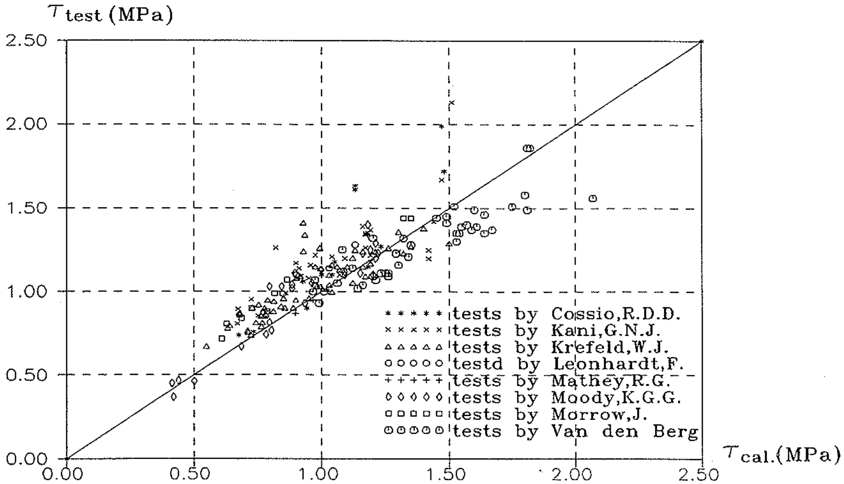


Fig.3.1.6.8. Comparison of the shear strength obtained by eq.(3.1.2.1) with x/h determined by eq.(3.1.6.2) and test results

In the above calculation, ν_s is put to 0.5. For comparison, a calculation with $\nu_s=0.7$ is also performed and the results are plotted in Fig.3.1.6.9. It can be seen that ν_s with a value between 0.5 and 0.7 will give better agreement with the test results. However, a refinement on this point must await more conclusive tests on this parameter. In the following $\nu_s=0.5$ will still be used.

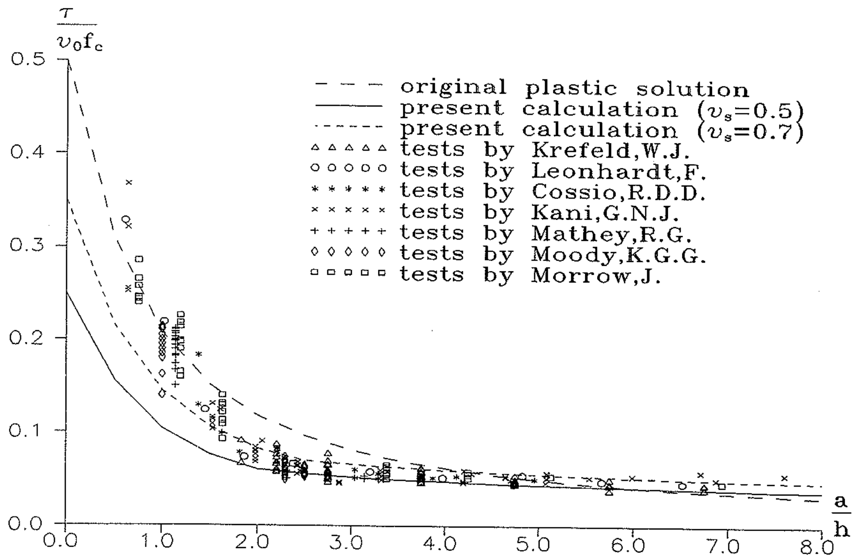
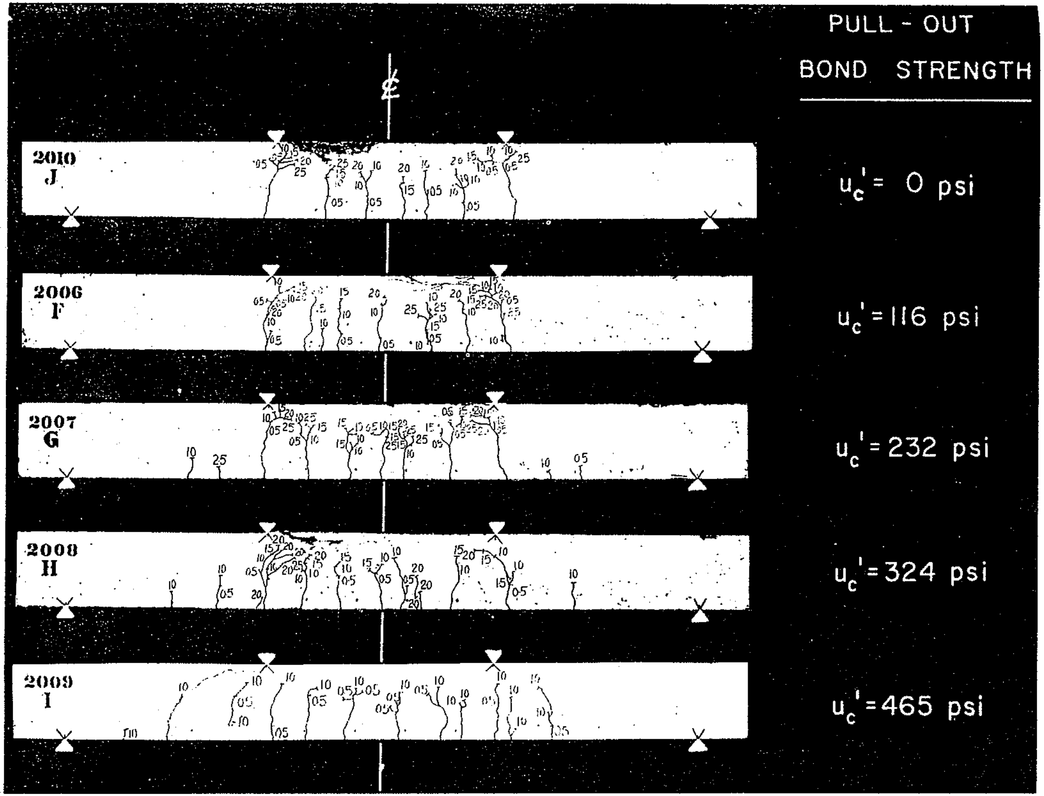


Fig 3.1.6.9 Comparison of theoretical shear strength using different values of ν_s and test results

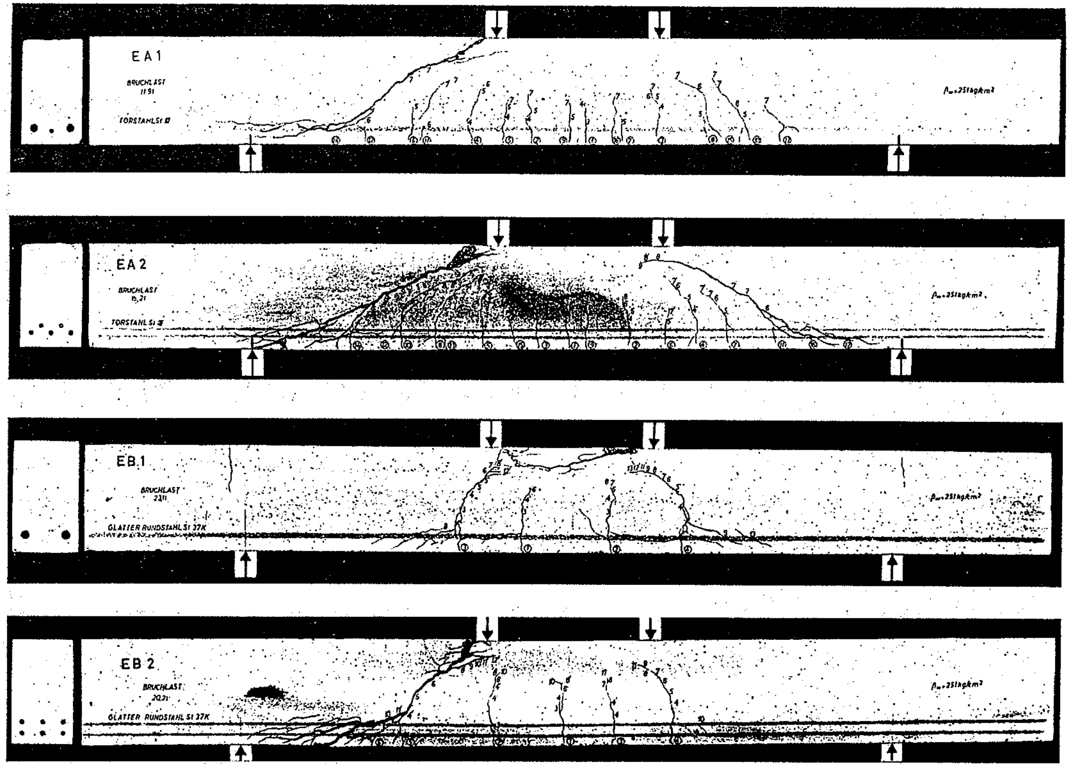
Beams with different bond conditions fail in different modes. We know that if the reinforcement and concrete is totally unbonded, the beam acts like a tied arch from beginning, and the failure is caused by the crushing of concrete in the compressive chord. No diagonal cracks occur before failure. The load carrying capacity in this case is higher than that with bond. This is confirmed by the tests performed by Kani [64,1] and Leonhardt [62,1]. The failure patterns with different bond conditions are shown in Fig.3.1.6.10.

In the plastic solution, the force in the longitudinal reinforcement is assumed constant, meaning no bond is considered. Therefore, the load carrying capacity



a). Tests by Kani [79,1]

Fig.3.1.6.10 Crack patterns of beams with different bond properties



b). Tests by Leonhardt & Walther [62,1]

Fig.3.1.6.10 (continued)

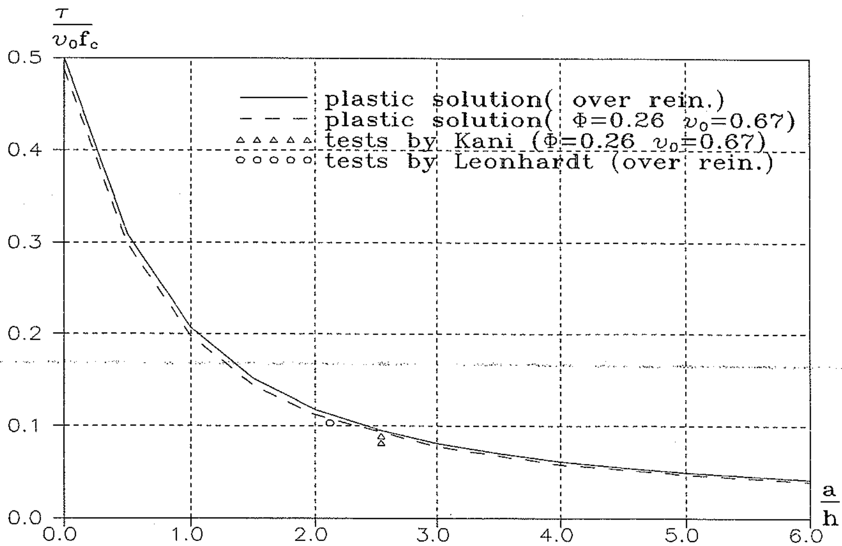


Fig.3.1.6.11 Shear strength compared with tests with no bond or little bond [62,1][79,1]

of beams with no bond or with little bond should correspond to the original plastic solution with the effectiveness factor ν_0. The comparison between tests and theoretical calculations is shown in Fig.3.1.6.11.

The number of beams tested without bond is scarce. From what we have found, it seems that the shear strength of beams with no bond or little bond corresponds very well to the original plastic solution.

It can be concluded from the above experimental verifications that the theory of the shear strength of cracked concrete beams is able to explain the shear failure mechanism and predict the shear capacity of the beams failed in shear with the formation of a critical diagonal crack in a very simple way.

3.2 Shear capacity of deep beams and corbels

Fig.3.2.1. shows a corbel with rectangular cross section. It can be treated in a way similar to what we have done in the former section.

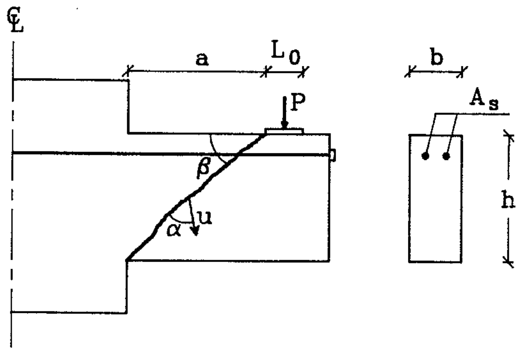


Fig.3.2.1 Corbel with rectangular cross section and a yield line

For deep beams and corbels, the shear span ratio is less than 1. No obvious critical diagonal cracks initiating from bending are observed. Therefore, the original plastic solution may be applied here.

The comparison between theoretical results and tests is depicted in Fig.3.2.2. As it has been stated in section 3.1.6, when $a/h \leq 0.75$, the effect of plane strain field will dominate the load carrying capacity and the load carrying capacity will be higher than that determined by the original plastic solution with the assumption of plane stress field.

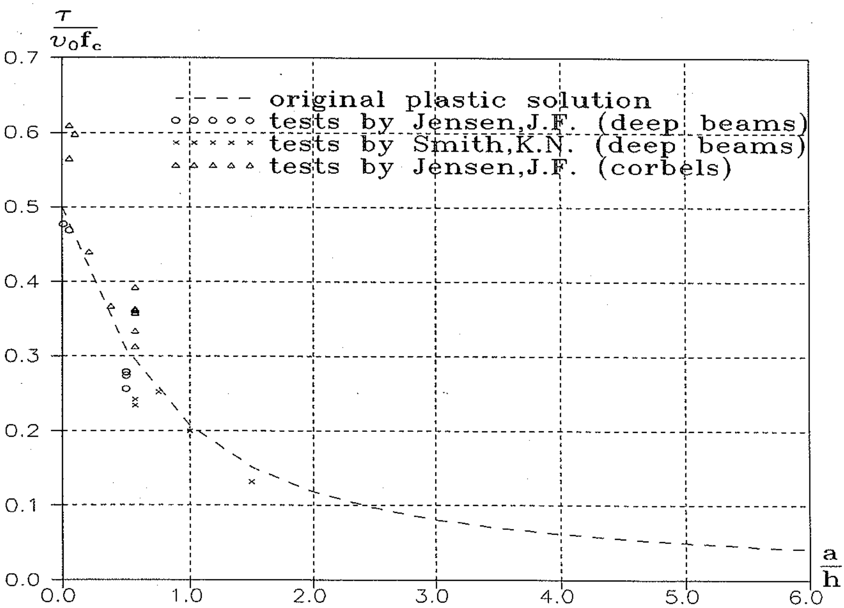
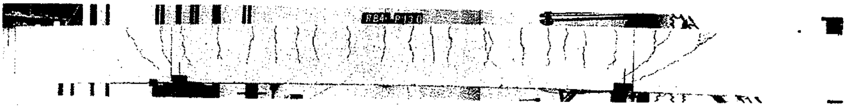


Fig.3.2.2 Shear strength of deep beams and corbels

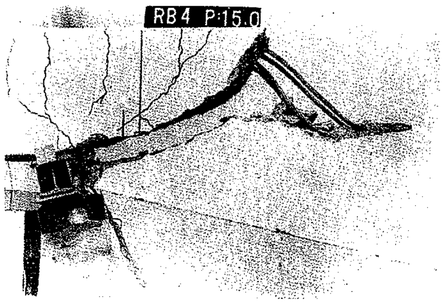
3.3 Shear capacity of prestressed beams under concentrated loading

Fig.3.3.1 shows a prestressed beam failed in shear with the crack patterns similar to those of the conventional reinforced concrete beams. In this section, only over reinforced beams with rectangular cross section are treated.

Fig.3.3.2 shows the shear span of a prestressed beam. The meaning of the symbols can be seen clearly in the figure. Here, F_{se} is the effective prestressing force in the reinforcement, which is supposed to be transferred to the concrete at the end of the beam.



Cracking at 130 kN



Failure at 140 kN

Fig.3.3.1 Crack pattern of a prestressed beam at failure [77,1]

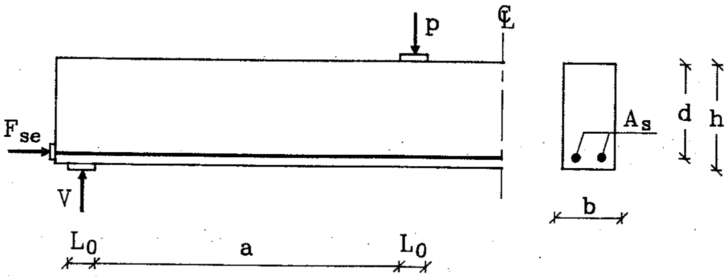


Fig.3.3.2 Shear span of a prestressed beam

3.3.1 Shear capacity according to plastic theory with a critical diagonal crack

According to the usual plastic theory, prestressing has no effect on the load carrying capacity since it creates an self equilibrated stress system, which is without influence on the load carrying capacity for a perfectly plastic structure.

However the prestressing affects the micro-cracking before failure and thereby the effectiveness factor. This is verified in [78,1] by choosing a proper function to calculate the effectiveness factor ν taking the prestressing force in the reinforcement into consideration.

For a prestressed beam failing in shear with a critical diagonal crack, the shear strength can be determined by eq.(3.1.2.1), taking into account that the value of ν must be different from that of a conventional reinforced concrete beam.

The failure mechanism presented here for beams failing in shear by formation of a critical diagonal crack implies that the shear force transferred through a crack after the formation of the crack should not be influenced by the prestressing force. Thus ν_s, which reflects the reduction of cohesion due to the cracking, should be the same as that of conventional beams.

It is a fact that for a reinforced concrete beam with normal prestressing reinforcement but no prestressing force, i.e. a beam acting as a conventional beam, the widths of cracks and the strains in the steel bars will be very large before failure due to the very low reinforcement ratio and high yield strength of the steel. Hence, the effective strength of the concrete will decrease, and the effectiveness factor will be smaller than that of a conventional beam with all other parameters being the same.

We know that prestressing is an effective way to prevent or delay the cracking. Thus the diagonal cracking load will be raised. The prestressing force will affect the redistribution of the stress field and the state of the micro-cracks. Consequently the value of ν_0 is depending on the prestressing force. The value of ν_0 for prestressed beams is taken as

$$\nu_0 = 1.2 f_1(f_c) \cdot f_2(h) \cdot f_3(\rho) \cdot f_4\left(\frac{\sigma_{se}}{f_c}\right) \tag{3.3.1.1}$$

where f_1, f_2, f_3 are the functions defined in eq.(2.5.2) to eq.(2.5.4). The function f_4 is dependent on the prestressing level. By trials and errors it takes the form

$$f_4\left(\frac{\sigma_{se}}{f_c}\right) = 1 + 2\frac{\sigma_{se}}{f_c} \tag{3.3.1.2}$$

Here $\sigma_{se} = F_{se}/bh$ is the average prestressing stress over the whole section, and σ_{se}/f_c is termed the level of prestressing.

The starting position of the critical diagonal crack will be determined in the following section.

3.3.2 Shear capacity determined by the diagonal cracking load

The same simplifications as for the conventional reinforced concrete beams are adopted here.

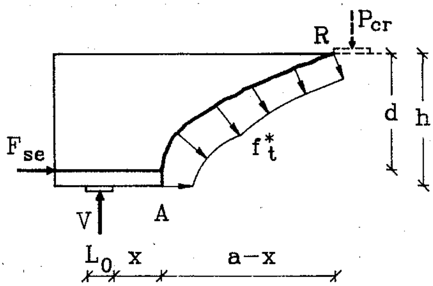


Fig.3.3.2.1 State of forces and stresses for maximum cracking moment in a prestressed beam

The moment equilibrium around point R leads to, see Fig.3.3.2.1,

$$V\cdot(\frac{1}{2}L_0+a)-F_{se}\cdot d=M_{cr} \qquad (3.3.2.1)$$

Here

$V=P_{cr}$

M_{cr} is determined by eq.(2.3.2)

The diagonal cracking load P_{cr} thus becomes

$$P_{cr}=\frac{1}{2}f_t^*\frac{bh(1+(\frac{a-x}{h})^2}{\frac{a}{h}+\frac{1}{2}\cdot\frac{L_0}{h}}+\frac{F_{se}\cdot\frac{d}{h}}{\frac{a}{h}+\frac{1}{2}\cdot\frac{L_0}{h}} \qquad (3.3.2.2)$$

Then the shear capacity determined by the diagonal cracking load $\tau_{cr}=P_{cr}/bh$ may be written as

$$\frac{\tau_{cr}}{\nu_0 f_c}=\frac{1}{2}\cdot\frac{f_t^*}{\nu_0 f_c}\cdot\frac{1+(\frac{a-x}{h})^2}{\frac{a}{h}+\frac{1}{2}\cdot\frac{L_0}{h}}+\frac{\sigma_{se}}{\nu_0 f_c}\cdot\frac{\frac{d}{h}}{\frac{a}{h}+\frac{1}{2}\cdot\frac{L_0}{h}} \qquad (3.3.2.3)$$

Here f_t^*, f_c^*, ν_s and L_0 are defined in the same way as for the conventional beams. The value of ν_0 is determined by eq.(3.3.1.1) and (3.3.1.2).

3.3.3 Starting position of the critical diagonal crack and shear capacity

As before, we assume that the ultimate load equals the diagonal cracking load. The equation $P_u = P_{cr}$ results in

$$\sqrt{1+\left(\frac{a-x}{h}\right)^2} - \frac{a-x}{h} = \frac{f_t^*}{f_c^*} \cdot \frac{1+\left(\frac{a-x}{h}\right)^2}{\frac{a}{h}+\frac{1}{2}\cdot\frac{L_0}{h}} + 2\frac{\sigma_{se}}{f_c^*} \cdot \frac{\frac{d}{h}}{\frac{a}{h}+\frac{1}{2}\cdot\frac{L_0}{h}} \qquad (3.3.3.1)$$

By solving eq.(3.3.3.1) numerically for different levels of prestressing, the starting position of the critical diagonal crack can be found, and consequently the shear capacity may be determined.

As we have done when treating conventional beams, the starting position of the critical diagonal crack x is put to zero when its calculated value becomes negative.

3.3.4 Parameter studies

In the parameter study, the default values are taken to be

$f_c = 35\text{MPa}$, $h = 30$ cm $\quad \rho = 0.6\%$ $\quad \nu_s = 0.5$

Values of the other parameters are determined in the same way as in section 3.1.5.

Logically, in prestressed beams the level of prestressing is the dominant parameter. Therefore, calculations with different values of σ_{se}/f_c are performed. The results are depicted in Fig.3.4.4.1.

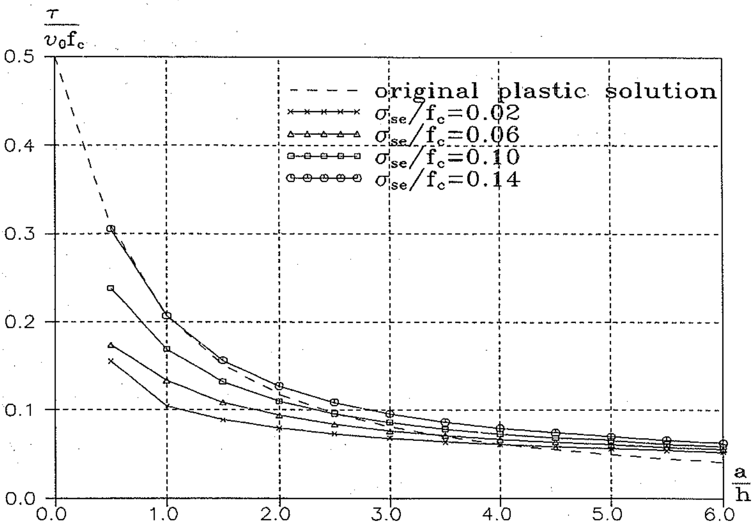


Fig.3.4.4.1 Shear strength for different levels of prestressing

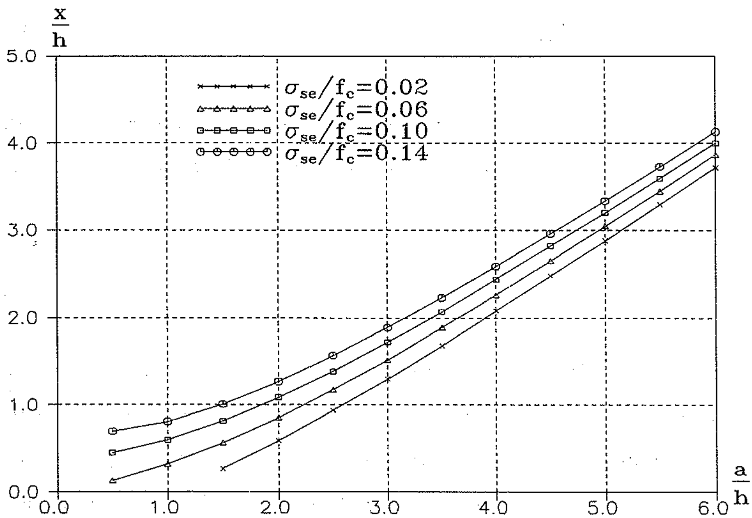


Fig.3.3.4.2 Value of x/h for different levels of prestressing

Obviously, different levels of prestressing will produce different cracking load. The higher the level is, the greater the diagonal cracking load becomes, and the higher the shear strength will be. The starting position of the critical diagonal crack moves farther away from the support. This is illustrated in Fig.3.3.4.1 and Fig.3.3.4.2.

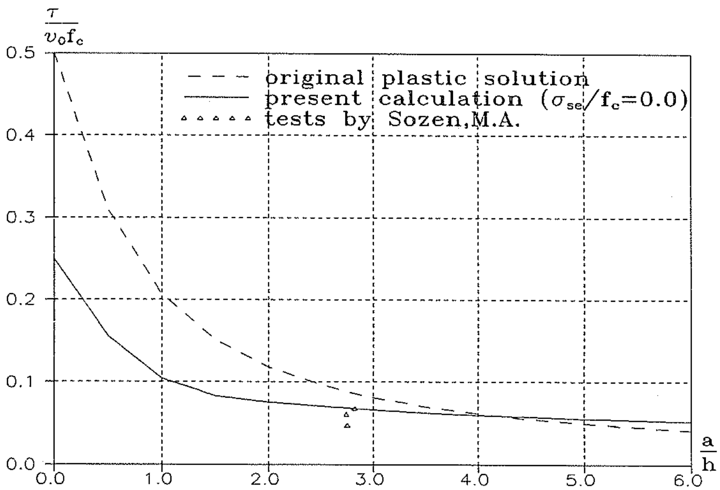
As we have mentioned in the analysis of conventional reinforced concrete beams, the calculation is not valid any more beyond the point where the curves of the original plastic solution and the present solution cross. When σ_{se}/f_c is around 0.14, the valid part of the curve of the present calculation almost coincides with the original plastic solution. This means that the present calculation is valid when the level of prestressing is less than about 0.14, and beyond this limit the shear strength will follow the original plastic solution. In this case no critical diagonal crack can form before shear failure.

3.3.5. Experimental verification

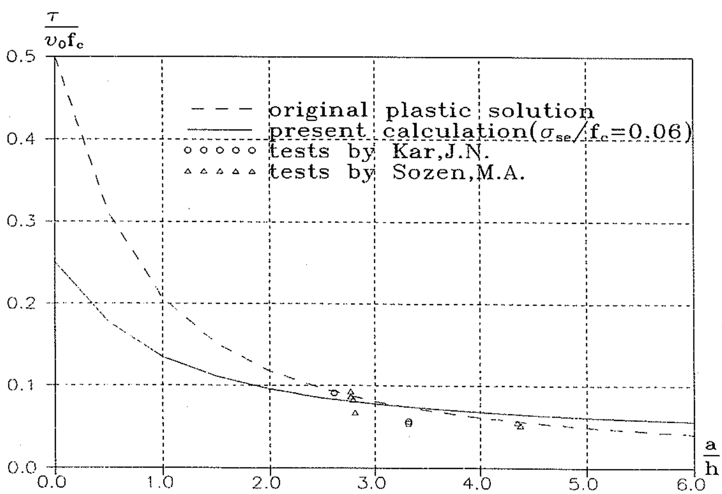
Only over reinforced concrete beams with rectangular cross section failing in shear are considered. Here the criterion for over reinforcement is

$$\Phi \geq \frac{1}{2}\nu_s \cdot \nu_0 \tag{3.3.5.1}$$

In Fig.3.3.5.1, the results of 54 beams from different tests series are plotted separately according to their prestressing levels. It can be concluded that the present calculations are in good agreement with the test results. When the prestressing level is above 0.14, the present calculation is not valid any more, and the test results will follow the original plastic solution.

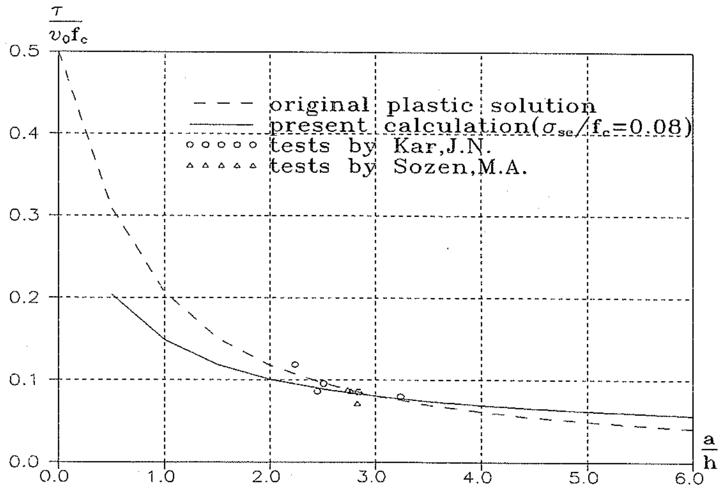


a). $\sigma_{se}/f_c \leq 0.01$

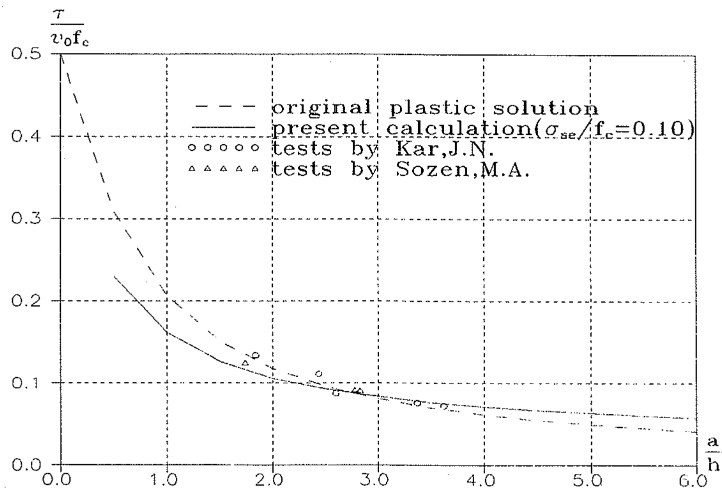


b). $0.05 \leq \sigma_{se}/f_c \leq 0.07$

Fig.3.3.5.1 Comparison of theoretical shear strength and test results at different levels of prestressing

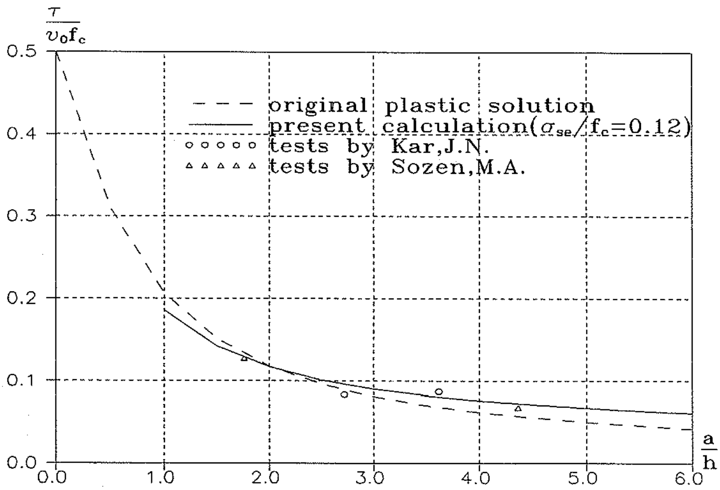


c). $0.07 < \sigma_{se}/f_c \leq 0.09$

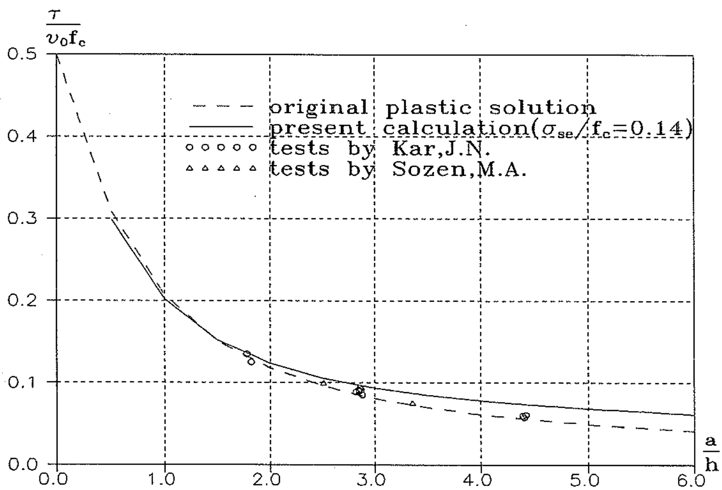


d). $0.09 < \sigma_{se}/f_c \leq 0.11$

Fig.3.3.5.1 (continued).



e). $0.11 < \sigma_{se}/f_c \leq 0.13$



f). $\sigma_{se}/f_c > 0.13$

Fig.3.3.5.1 (continued).

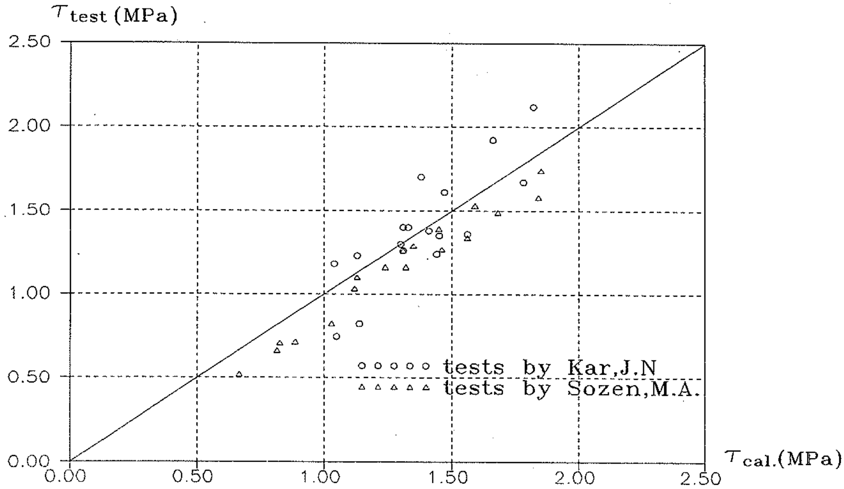


Fig.3.3.5.2 Comparison of theoretical calculations and test results

Fig.3.3.5.2 shows the comparison between theoretical calculation and 35 test results from [59,1] [69,1] with $0.05 \leq \sigma_{se}/f_c \leq 0.14$. The mean value of the ratio $\tau_{test}/\tau_{cal.}$ is 0.94, the standard deviation is 0.13, and the coefficient of variation is 0.14.

3.4. Shear capacity of conventional beams subjected to uniform loading

Not many test series of uniformly loaded beams are reported. Also in the plastic theory, uniformly loaded beams are not treated as thoroughly as the beams under concentrated loads.

In [77,1] it was found that the beams in average carried the load corresponding to the bending failure load. But Fig.3.4.1 shows clearly that a shear failure mechanism with the formation of a critical diagonal crack exists also for beams subjected to uniformly distributed load. Here for convenience we term the shear span a as half the clear span of the beam.

3.4.1. Shear capacity according to the usual plastic theory

Only over reinforced concrete beams with rectangular cross section are considered. The shear span of a beam is shown in Fig.3.4.1.1, where the meanings of the symbols can be seen clearly. The load is characterized by the normal pressure p on the top face.

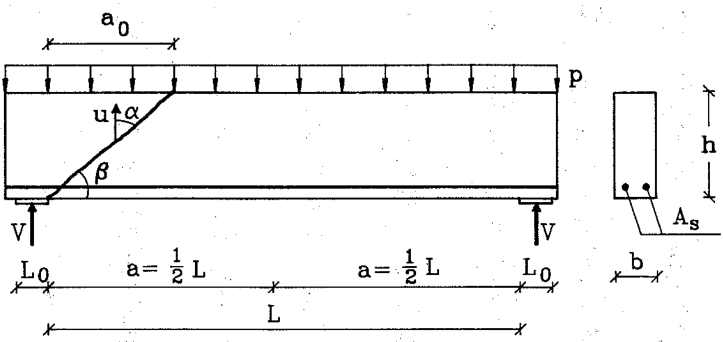
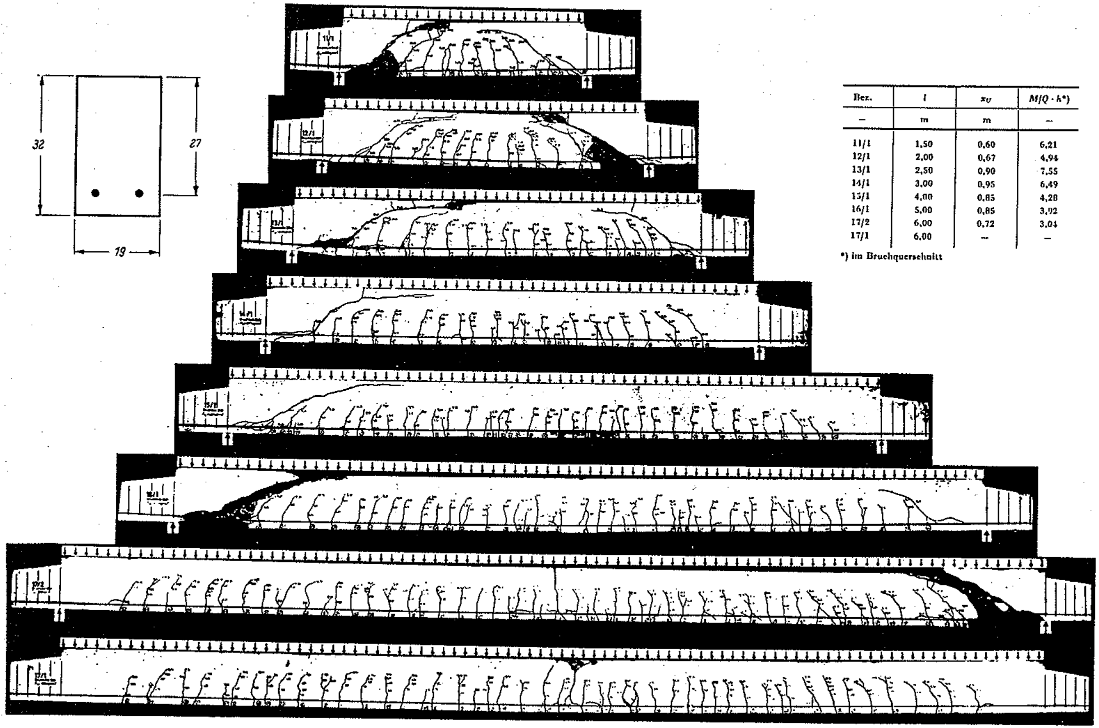


Fig.3.4.1.1 Shear span of a beam and a yield line in an upper bound solution of the usual plastic theory

We consider an upper bound solution corresponding to a straight yield line starting from the edge of the support platen. Since the beam is over reinforced, the relative displacement u is perpendicular to the reinforcement forming an angle $\alpha = 90° - \beta$ with the yield line. The internal work dissipated by this failure



Bez.	l	x_U	$M/(Q \cdot h^*)$
–	m	m	–
11/1	1,50	0,60	6,21
12/1	2,00	0,67	4,94
13/1	2,50	0,90	7,55
14/1	3,00	0,95	6,49
15/1	4,00	0,85	4,28
16/1	5,00	0,85	3,92
17/2	6,00	0,72	3,04
17/1	6,00	–	–

*) im Bruchquerschnitt

Fig.3.4.1 Crack patterns of beams at shear failure under uniform loading tested by Leonhardt & Walther [62,1]

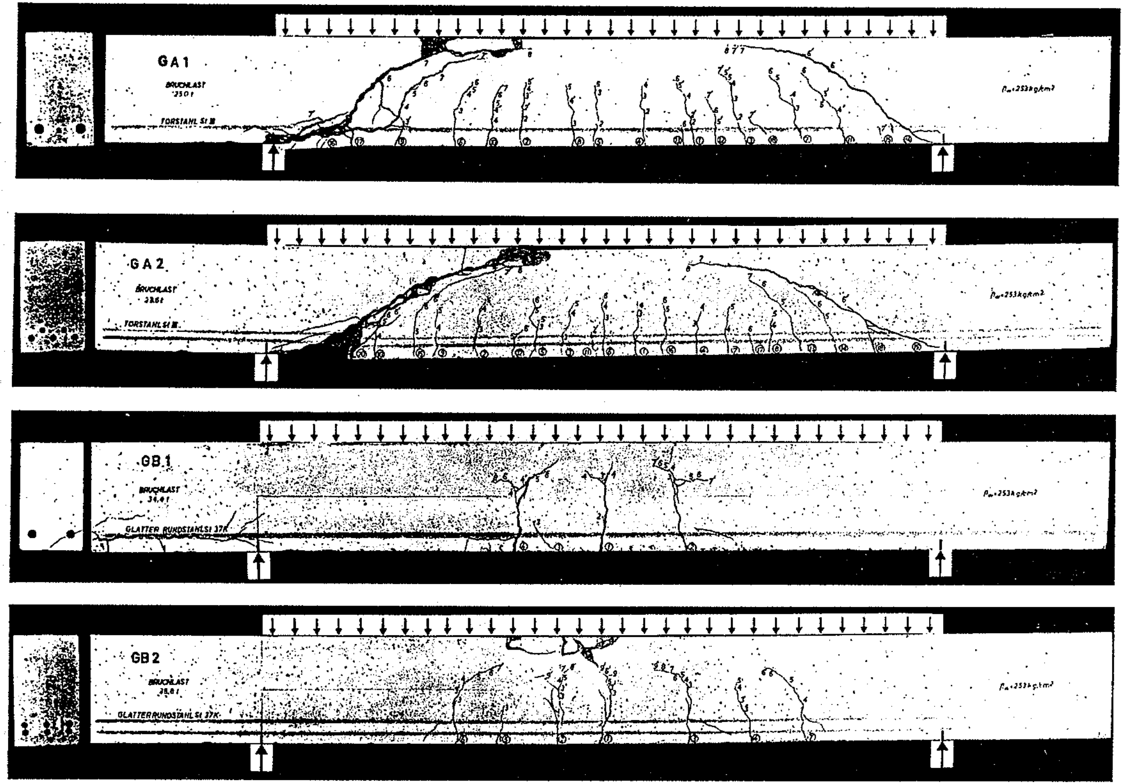


Fig.3.4.2 Crack patterns of beams under uniform loading with different bond properties [62,1]

mechanism is

$$W_I = \frac{1}{2} f_{c0}^* b(1 - \cos\beta) \frac{h}{\sin\beta} \cdot u \qquad (3.4.1.1)$$

$$\cot\beta = \frac{a_0}{h}$$

The external work is

$$W_E = R \cdot u - pb(a_0 + L_0) \cdot u$$

Since

$$V = pb(a + L_0)$$

then

$$W_E = pb(a - a_0) \cdot u \qquad (3.4.1.2)$$

Here a_0 is the horizontal distance between the upper end of the yield line and the edge of the support platen (see Fig.3.4.1.1).

The work equation $W_I = W_E$ yields

$$p = \frac{1}{2} f_{c0}^* \frac{\sqrt{1 + \left(\frac{a_0}{h}\right)^2} - \frac{a_0}{h}}{\frac{a}{h} - \frac{a_0}{h}} \qquad (3.4.1.3)$$

Minimizing eq.(3.4.1.3) with respect to a_0 results in the lowest upper bound solution

$$p_{min}=f_{c0}^{*}\frac{1}{1+(\frac{a}{h})^2} \qquad (3.4.1.4)$$

$$\frac{a_0}{h}=\frac{(\frac{a}{h})^2-1}{2(\frac{a}{h})} \qquad (3.4.1.5)$$

We take the average shear stress at the edge of the support platen as the nominal shear stress $\tau=V'/bh$. Thus the shear strength τ_0 determined by this shear failure mechanism becomes

$$\frac{\tau_0}{f_{c0}^{*}}=\frac{\frac{a}{h}}{1+(\frac{a}{h})^2} \qquad (3.4.1.6)$$

since

$$V'=p_{min}ba.$$

The shear strength is normalized by being divided by the effective compressive strength of micro-cracked concrete to facilitate the comparison with tests.

Eq.(3.4.1.4) to (3.4.1.6) are valid for $a/h \geq 1$.

As before f_{c0}^{*} is determined by $f_{c0}^{*}=\nu_0 f_c$. The value of ν_0 would be different from the one valid for beams under concentrated loading due to the differences in the state of micro cracks and stress redistribution.

From the comparison with tests and statistical analysis, the value of ν_0 is taken to be

$$\nu_0 = 1.2 f_1(f_c) \cdot f_2(h) \cdot f_3(\rho) \qquad (3.4.1.7)$$

Here f_1,f_2 and f_3 are determined by eq.(2.5.2) to eq.(2.5.4).

In the following sections, the solution presented here is referred to as the original plastic solution.

It should be noted that the upper bound solution deducted here is not an exact plastic solution. The exact plastic solution found in [81,1] corresponds to a load carrying capacity given by eq.(3.4.1.4) with the denominator $1+(a/h)^2$ replaced by $1+2(a/h)^2$. However this solution is not interesting here because no shear failure mechanism may be found to be related to it. From the pictures in Fig.3.4.1.1 it is obvious that a shear failure similar to the shear failure mechanism treated here exists.

3.4.2. Shear capacity according to plastic theory with a critical diagonal crack

The picture from tests in Fig.3.4.1 shows that the final diagonal crack tends to end around the middle of the shear span. This point is close to the point where the yield line ends, according to the original plastic solution. As we have done for the beams under concentrated loading, where the end of the critical diagonal crack coincides with the end of the straight yield line determined from the original plastic solution, similarly the end of the critical diagonal crack here is put to the end of the yield line found in the original plastic solution which is at a distance a_0 from the edge of the support platen. The value of a_0 is determined by eq.(3.4.1.5).

The same assumptions as in section 3.1 are applied here. A straight yield line is assumed to follow a simplified straight critical diagonal crack as shown in Fig.3.4.2.1 where the starting position of the critical diagonal crack is determined by x.

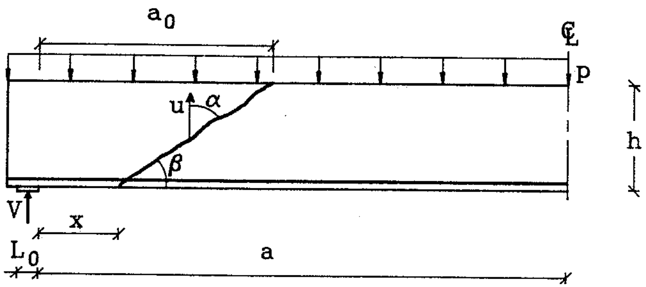


Fig.3.4.2.1 Shear failure mechanism with a critical diagonal crack

In the over reinforced case, the internal work is

$$W_I = \frac{1}{2} f_c^* b(1-\cos\beta)\frac{h}{\sin\beta} \cdot u \qquad (3.4.2.1)$$

Since

$$\cot\beta = \frac{a_0 - x}{h}$$

thus the internal work becomes

$$W_I = \frac{1}{2} f_c^* bh\left(\sqrt{1+\left(\frac{a_0-x}{h}\right)^2} - \frac{a_0-x}{h}\right)\cdot u \qquad (3.4.2.2)$$

The external work W_E is the same as given by eq.(3.4.1.2).

The work equation $W_I=W_E$ results in the following ultimate load p_u and shear strength $\tau_u=V'_u/bh=p_u a/h$

$$p_u=\frac{1}{2}f_c^* \frac{\sqrt{1+\left(\frac{a_0-x}{h}\right)^2}-\frac{a_0-x}{h}}{\frac{a}{h}-\frac{a_0}{h}} \quad (3.4.2.3)$$

$$\frac{\tau_u}{\nu_0 f_c}=\frac{1}{2}\nu_s \frac{\left(\sqrt{1+\left(\frac{a_0-x}{h}\right)^2}-\frac{a_0-x}{h}\right)\cdot\frac{a}{h}}{\frac{a}{h}-\frac{a_0}{h}} \quad (3.4.2.4)$$

Here $f_c^*=\nu f_c$ and $\nu=\nu_s \cdot \nu_0$.

According to the shear failure mechanism presented here, the factor ν_s should not be affected by the loading type. Therefore the value of ν_s is put to 0.5 in the calculation.

3.4.3 Shear capacity determined by the diagonal cracking load

Same assumptions and simplifications as for beams under concentrated loading are applied here.

The moment equilibrium about point R yields, see Fig.3.4.3.1,

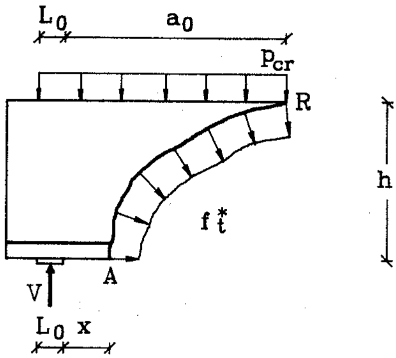


Fig.3.4.3.1 State of forces and stresses for maximum cracking moment

$$V\cdot(a_0+\frac{1}{2}L_0)-\frac{1}{2}pb\cdot(a_0+L_0)^2=M_{cr} \tag{3.4.3.1}$$

where

$$V=pb(a+L_0)$$

$$L_{AR}=\sqrt{h^2+(a_0-x)^2}$$

M_{cr} is determined by eq.(2.3.4)

The diagonal cracking load becomes

$$p_{cr}=\frac{1}{2}f_t^*\frac{1+(\frac{a_0}{h}-\frac{x}{h})^2}{(\frac{a}{h}+\frac{L_0}{h})\cdot(\frac{a_0}{h}+\frac{1}{2}\cdot\frac{L_0}{h})-\frac{1}{2}(\frac{a_0}{h}+\frac{L_0}{h})^2} \tag{3.4.3.2}$$

The shear capacity determined by the diagonal cracking load is

$$\tau_{cr}=\frac{1}{2}f_t^*\frac{\left(1+\left(\frac{a_0}{h}-\frac{x}{h}\right)^2\right)\cdot\frac{a}{h}}{\left(\frac{a}{h}+\frac{L_0}{h}\right)\cdot\left(\frac{a_0}{h}+\frac{1}{2}\cdot\frac{L_0}{h}\right)-\frac{1}{2}\left(\frac{a_0}{h}+\frac{L_0}{h}\right)^2} \tag{3.4.3.3}$$

The value of f_t^* is determined in the same way as before.

3.4.4 Starting position of the critical diagonal crack

The specimen fails in shear when the ultimate plastic load for a diagonal crack reaches the diagonal cracking load. Putting $p_u=p_{cr}$ results in

$$f_c^*\cdot\frac{\sqrt{1+\left(\frac{a_0-x}{h}\right)^2}-\frac{a_0-x}{h}}{\frac{a}{h}-\frac{a_0}{h}}=f_t^*\cdot\frac{1+\left(\frac{a_0-x}{h}\right)^2}{\left(\frac{a}{h}+\frac{L_0}{h}\right)\cdot\left(\frac{a_0}{h}+\frac{1}{2}\cdot\frac{L_0}{h}\right)-\frac{1}{2}\left(\frac{a_0}{h}+\frac{L_0}{h}\right)^2} \tag{3.4.4.1}$$

The value of x for a given shear span ratio a/h is, as before, determined numerically.

It is found that x becomes negative when $a/h<4$.

3.4.5 Experimental verification

Only over reinforced beams with rectangular cross section failing in shear are treated. The criterion for over reinforcement is

$$\Phi \geq \frac{1}{2} \nu_0 \cdot \nu_s \qquad (3.4.5.1)$$

The default values of variables in the theoretical calculation are set to

$f_c = 25MPa \quad h = 30cm \quad \rho = 2\% \quad \nu_s = 0.5$

The length of load and support platens L_0 is taken from the test reports or determined from the stress field in the lower bound solution of the plastic theory. Details see [81,1].

The comparison of theoretical shear strength and test results is depicted in Fig.3.4.5.1.

It can be seen that:

(1). When $a/h \geq 4$, the agreement between tests and present calculation is rather good. And from the pictures shown in Fig.3.4.1 we can see that a critical diagonal crack really is forming in this range of shear spans.

(2). When $a/h < 4$, and consequently $a_0/h < 1.9$, the starting position of the critical diagonal crack determined from the calculation is negative. This value of a_0/h is close to the value of a/h leading to negative value of x for concentrated loaded beams. The discussion in section 3.1.6 for concentrated loaded beams with negative x values may be transferred to this case with slight modifications.

The discussion related to plane strain conditions in section 3.1.6 for short shear spans is valid for uniformly loaded beams as well. Here the angle α between the relative displacement and the yield line becomes less than φ when

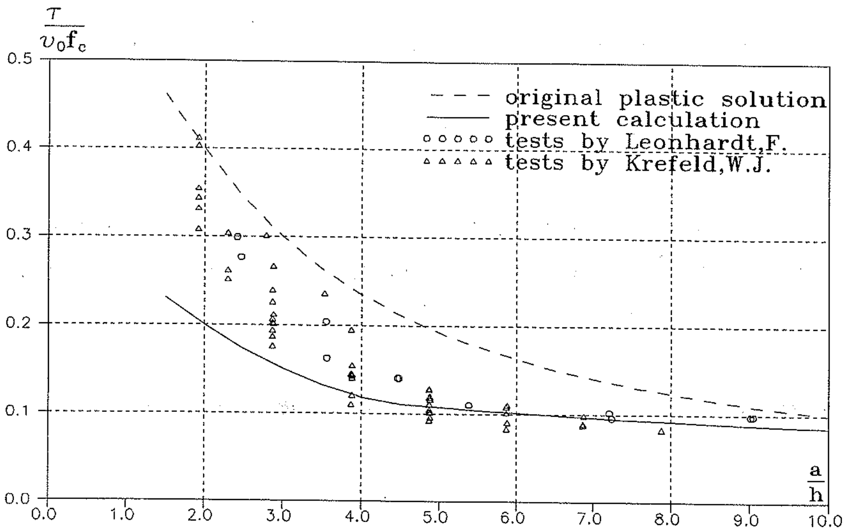


Fig.3.4.5.1 Comparison of theoretical shear strength and test results [62,1][66,1]

$a_0/h < 0.75$, corresponding to $a/h < 2$ (deducted from eq.(3.4.1.5)).

Fig.3.4.5.2 shows the comparison of shear strength determined by the theoretical calculations and 29 test results with $a/h \geq 4$. The mean value of the ratio $\tau_{test}/\tau_{cal.}$ is 1.11, the standard deviation is 0.11, and the coefficient of variation is 0.10.

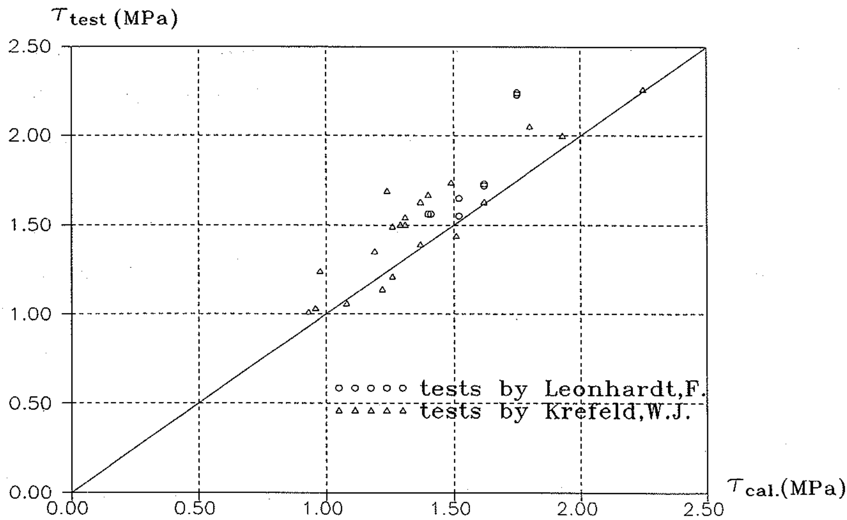


Fig.3.4.5.2 Comparison of theoretical calculation and test results for $a/h \geq 4$

IV. CONCLUSION

In this report a new theory which deals with the shear capacity of non shear reinforced concrete beams is proposed. The theory is based on the plastic theory of concrete. The new theory gives a rational explanation for the shear failure of concrete beams featured by the formation of a critical diagonal crack. The principle of the shear failure mechanism suggested in this paper may be applied to both conventional reinforced concrete beams and prestressed reinforced concrete beams subjected to different loading types.

In the theory it is assumed that the cracking introduces potential yield lines which may be more dangerous than those found in the usual plastic theory. By assuming that the critical diagonal crack at shear failure is transformed into a yield line, the starting position of this crack can be found, and consequently the shear strength may be determined.

It is found that the theoretical results are in good agreement with the test results.

In the theory of plasticity, the strength of concrete needs to be modified by an effectiveness factor in order to get agreement with test results. In the new theory, the distinction between the effectiveness factor for micro-cracked concrete and the effectiveness factor for macro-cracked concrete is made. The effectiveness factor for micro-cracked concrete ν_0 is related to the uniaxial compressive strength of the concrete, the reinforcement ratio, the depth of the section and the loading type. The effectiveness factor for macro-cracked concrete ν is obtained by multiplying ν_0 with a sliding reduction factor ν_s which reflects the reduction of the sliding strength due to cracking.

The comparison between theoretical results and test results reveals that the value of ν_0 is different for different loading types due to the differences in the state of the micro-cracks, the stress field and the redistribution of the stresses.

For conventional beams subjected to concentrated loading, the starting position of the critical diagonal crack may be estimated approximately by a linear function of the shear span ratio, and the shear strength may be easily determined by the plastic solution.

Similar relationships may be found for conventional beams subjected to uniformly distributed loading and for prestressed beams.

It has to be admitted that some simplifications made in this paper are rather rough. More research is needed in the following fields:

1. **The sliding reduction factor ν_s**. In the paper ν_s is taken as a constant 0.5. However it is affected by many factors such as the width of cracks, the size and strength of aggregates, the confining conditions in the crack, etc.

2. **The effectiveness factor for micro-cracked concrete and for macro-cracked concrete**. A more rational way to distinguish between the effectiveness factor of uncracked or micro-cracked concrete and that of cracked concrete may lead to a better understanding of the mechanism of shear failure.

3. **The cracking load**. In this paper, the diagonal cracking load is determined under the assumption of a plastic equivalent stress distribution similar to that made in the case of bending. More accurate solutions would need a fracture

mechanics treatment.

4. **The ultimate load**. In this paper the calculation of the ultimate load according to the plastic theory corresponding to a curved yield line coinciding with a curved crack is simplified by using a straight line approximation. When more accurate values of ν_s is known and a more accurate way of estimating the diagonal cracking load has been set up, then more accurate ultimate load values may be easily found for a curved yield line.

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APPENDIX. RECOMMENDATIONS FOR PRACTICE

From the comparisons of the theoretical shear strength and experimental results presented in this paper, the following recommendations may be made for the determination of the shear capacity of non shear reinforced concrete beams in practical design.

1. Shear strength of conventional non shear reinforced beams subjected to concentrated loading

For conventional non shear reinforced beams subjected to concentrated loading with shear span ratio a/h between 2 and 5, the normalized shear strength (being divided by the effective compressive strength of uncracked or micro-cracked concrete $\nu_0 f_c$) as a function of a/h is depicted in Fig.A1.1.

Approximately, the shear strength may be expressed by

$$\frac{\tau}{\nu_0 f_c} = 0.055 - 0.004\left(\frac{a}{h} - 2\right) \qquad \text{(A1.1)}$$

Here

$$\nu_0 = 1.6 f_1(f_c) \cdot f_2(h) \cdot f_3(\rho) \qquad \text{(A1.2)}$$

$$f_1 = \frac{3.5}{\sqrt{f_c}} \qquad (f_c \text{ in MPa}) \qquad \text{(A1.3)}$$

$$f_2 = 0.27\left(1 + \frac{1}{\sqrt{h}}\right) \qquad (h \text{ in m}) \qquad \text{(A1.4)}$$

$$f_3 = 0.15\rho + 0.58 \qquad (\rho \text{ in } \%) \qquad \text{(A1.5)}$$

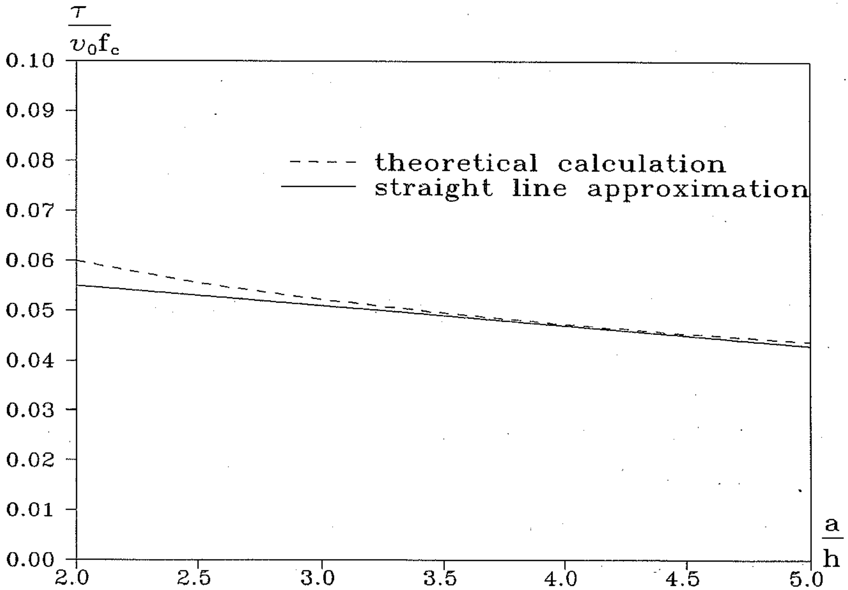

Fig.A1.1 Shear strength versus a/h ($2 \leq a/h \leq 5$)

Fig.A1.2 shows the comparison of the shear strength determined by eq.(A1.1) and test results for $2 \leq a/h \leq 5$. The mean value of the ratio $\tau_{test}/\tau_{cal.}$ is 1.11, and the standard deviation is 0.16.

Conservatively, the shear strength may be taken as

$$\frac{\tau}{\nu_0 f_c} = 0.045 \qquad \text{(A1.6)}$$

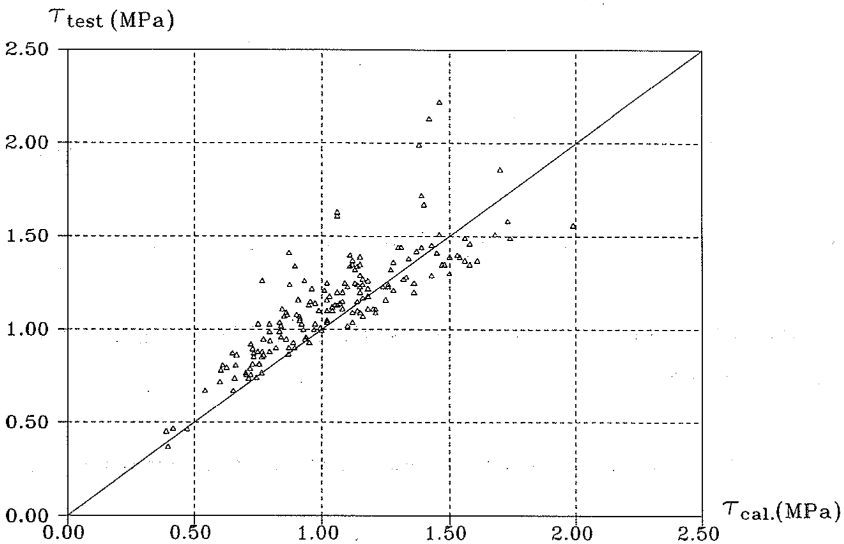

Fig.A1.2 Comparison of calculated shear strength and test results from [66,1][62,1][60,2][64,1][63,1][54,2][57,1][62,2] for $2 \leq a/h \leq 5$

The starting position of the critical diagonal crack x may be predicted approximately by

$$\frac{x}{h}=0.74\left(\frac{a}{h}-2\right) \qquad \left(\frac{a}{h}\geq 2\right) \qquad \text{(A1.7)}$$

At present, for beams with shear span ratio $0.75<a/h<2$, the shear strength may be determined by

$$\frac{\tau}{\nu_0 f_c} = \frac{1}{2}\nu_s\left(\sqrt{1+\left(\frac{a}{h}\right)^2} - \frac{a}{h}\right) \qquad \text{(A1.8)}$$

Here

$$\nu_s = 0.5 \qquad \text{(A1.9)}$$

For beams with $a/h < 0.75$ the original plastic solution may be used.

2. Shear strength of non shear reinforced prestressed beams subjected to concentrated loading

The effectiveness factor for uncracked prestressed concrete beams is taken to be

$$\nu_0 = 1.2 f_1(f_c) \cdot f_2(h) \cdot f_3(\rho) \cdot f_4\left(\frac{\sigma_{se}}{f_c}\right) \qquad \text{(A2.1)}$$

Here

f_1, f_2, f_3 are determined by eq.(A1.2)-(A1.4)

$$f_4 = 1 + 2\frac{\sigma_{se}}{f_c} \qquad \text{(A2.2)}$$

$$\sigma_{se} = \frac{F_{se}}{bh} \qquad \text{(A2.3)}$$

In practice it would be convenient to remove the effect of prestressing stress from the value of the effectiveness factor. Instead the effectiveness factor for conventional beams determined by eq.(A1.1) is adopted. The effect of the prestressing is taken into account by other parameters. The normalized shear strength as a function of the prestressing level for different shear span ratio a/h

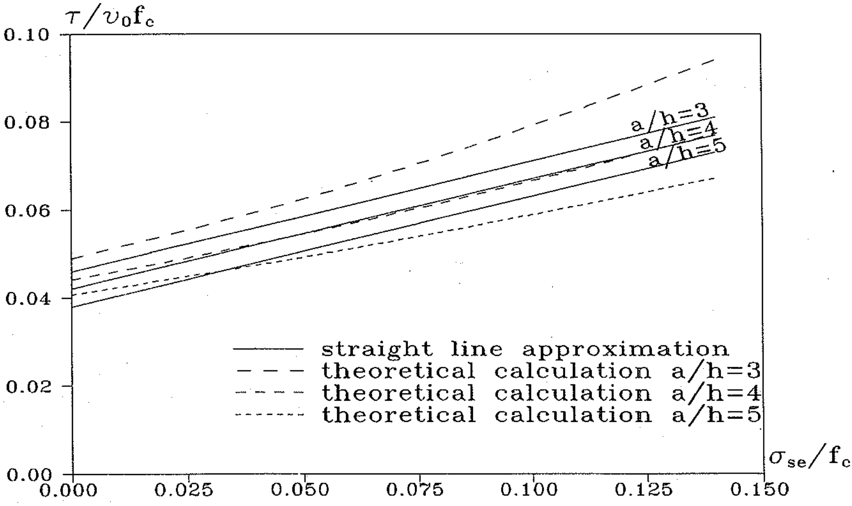

Fig.A2.1 Shear strength versus σ_{se}/f_c for different a/h

is depicted in Fig.A2.1. Approximately, the function may be expressed as:

$$\frac{\tau}{\nu_0 f_c}=0.25\frac{\sigma_{se}}{f_c}+0.05-0.004\left(\frac{a}{h}-2\right) \qquad \left(\frac{\sigma_{se}}{f_c}\leq 0.14 \text{ and } 2\leq\frac{a}{h}\leq 5\right) \tag{A2.4}$$

Here

$$\nu_0=1.6f_1(f_c)\cdot f_2(h)\cdot f_3(\rho)$$

Fig.A2.2 shows the comparison of shear strength determined by eq.(A2.4) and test results. The mean value of the ratio $\tau_{test}/\tau_{cal.}$ is 1.02, the standard deviation is 0.21.

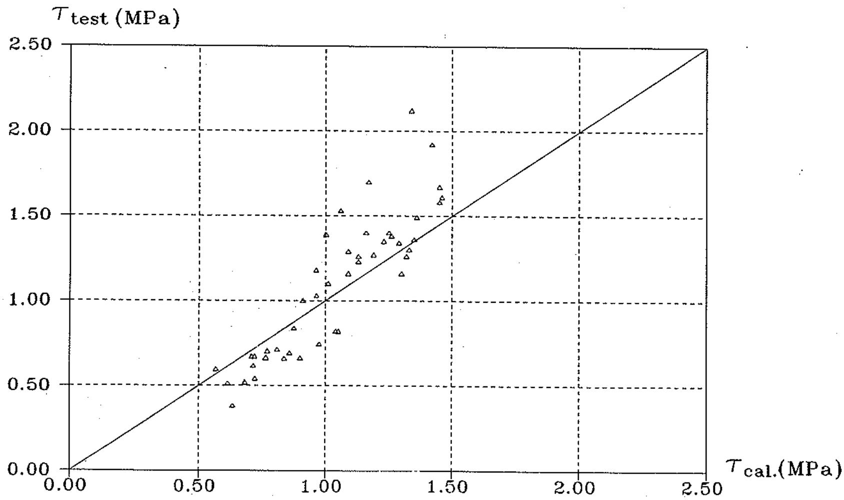

Fig.A2.2 Comparison of calculated shear strength and test results from [59,1][69,1] with $\sigma_{se}/f_c \leq 0.14$ and $2 \leq a/h \leq 5$

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